Lectures 16: Geodesics

Disclaimer. As we have a textbook, this lecture note is for guidance and supplement only. It should not be relied on when preparing for exams.

In this lecture we study geodesics, a special kind of curve on a surface, characterized by unit tangent vectors being parallel along the curve.

The required textbook sections are §7.4, §9.1–9.4. The optional sections are §9.5

I try my best to make the examples in this note different from examples in the textbook. Please read the textbook carefully and try your hands on the exercises. During this please don't hesitate to contact me if you have any questions.

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1. An example for calculation of Γ_{ij}^k.

We first give one more example on how to calculate Γ_{ij}^k.

Example 1. Let S be the unit sphere $(\cos u \cos v, \cos u \sin v, \sin u)$. We calculate Γ_{ij}^k.

i. Preparation.

We have

$$\sigma_u = (-\sin u \cos v, -\sin u \sin v, \cos u), \quad \sigma_v = (-\cos u \sin v, \cos u \cos v, 0), \tag{1}$$

$$N = (\cos u \cos v, \cos u \sin v, \sin u) \tag{2}$$

$$\begin{aligned} \sigma_{uu} &= (-\cos u \cos v, -\cos u \sin v, -\sin u), &(3)\\ \sigma_{uv} &= (\sin u \sin v, -\sin u \cos v, 0), &(4)\\ \sigma_{vv} &= (-\cos u \cos v, -\cos u \sin v, 0). &(5)\end{aligned}$$

ii. Solving for Γ_{ij}^k.

- Γ_{11}^k. We need to solve

$$\begin{pmatrix} -\cos u \cos v \\ -\cos u \sin v \\ -\sin u \end{pmatrix} = \Gamma_{11}^1 \begin{pmatrix} -\sin u \cos v \\ -\sin u \sin v \\ \cos u \end{pmatrix} + \Gamma_{11}^2 \begin{pmatrix} -\cos u \sin v \\ \cos u \cos v \\ 0 \end{pmatrix} + \mathbb{L} \begin{pmatrix} \cos u \cos v \\ \cos u \sin v \\ \sin u \end{pmatrix}. \tag{6}$$

  It is easy to see that $\Gamma_{11}^1 = \Gamma_{11}^2 = 0, \mathbb{L} = -1$.

- Γ_{12}^k. We need to solve

$$\begin{pmatrix} \sin u \sin v \\ -\sin u \cos v \\ 0 \end{pmatrix} = \Gamma_{12}^1 \begin{pmatrix} -\sin u \cos v \\ -\sin u \sin v \\ \cos u \end{pmatrix} + \Gamma_{12}^2 \begin{pmatrix} -\cos u \sin v \\ \cos u \cos v \\ 0 \end{pmatrix} + \mathbb{M} \begin{pmatrix} \cos u \cos v \\ \cos u \sin v \\ \sin u \end{pmatrix}. \tag{7}$$

  To solve this we first notice that

$$\mathbb{M} = \begin{pmatrix} \sin u \sin v \\ -\sin u \cos v \\ 0 \end{pmatrix} \cdot \begin{pmatrix} \cos u \cos v \\ \cos u \sin v \\ \sin u \end{pmatrix} = 0. \tag{8}$$

  A moment's inspection to

$$\begin{pmatrix} \sin u \sin v \\ -\sin u \cos v \\ 0 \end{pmatrix} = \Gamma_{12}^1 \begin{pmatrix} -\sin u \cos v \\ -\sin u \sin v \\ \cos u \end{pmatrix} + \Gamma_{12}^2 \begin{pmatrix} -\cos u \sin v \\ \cos u \cos v \\ 0 \end{pmatrix} \tag{9}$$

  reveals that $\Gamma_{12}^1 = 0, \Gamma_{12}^2 = -\tan u$.

- Γ_{22}^k. We need to solve

$$\begin{pmatrix} -\cos u \cos v \\ -\cos u \sin v \\ 0 \end{pmatrix} = \Gamma_{22}^1 \begin{pmatrix} -\sin u \cos v \\ -\sin u \sin v \\ \cos u \end{pmatrix} + \Gamma_{22}^2 \begin{pmatrix} -\cos u \sin v \\ \cos u \cos v \\ 0 \end{pmatrix} + \mathbb{N} \begin{pmatrix} \cos u \cos v \\ \cos u \sin v \\ \sin u \end{pmatrix}. \tag{10}$$

  This time we see that

$$\mathbb{N} = \begin{pmatrix} -\cos u \cos v \\ -\cos u \sin v \\ 0 \end{pmatrix} \cdot \begin{pmatrix} \cos u \cos v \\ \cos u \sin v \\ \sin u \end{pmatrix} = -\cos^2 u. \tag{11}$$

  Now the equation for the third component becomes

$$0 = \Gamma_{22}^1 \cos u + (-\cos^2 u) \sin u \tag{12}$$

  which gives

$$\Gamma_{22}^1 = \sin u \cos u. \tag{13}$$

  Substituting the values of $\mathbb{N}$ and Γ_{22}^1 into (10) gives $\Gamma_{22}^2 = 0$.

The parallel transport equations now become

$$\dot{\alpha} + (\sin u \cos u)\, \dot{v}\, \beta = 0, \qquad \dot{\beta} - (\tan u)\, \dot{v}\, \alpha = 0. \tag{14}$$

Thus w is parallel along γ if and only if (14) holds.

Now notice that unless $\sin u = 0$, that is γ is the big circle, the solution does not satisfy $\dot{\alpha} = \dot{\beta} = 0$.

Remark 2. Feel free to apply the formulas . We have

$$\mathbb{E} = 1, \qquad \mathbb{F} = 0, \qquad \mathbb{G} = \cos^2 u. \tag{15}$$

This leads to

$$\begin{aligned}
\Gamma_{11}^1 &= \frac{\mathbb{G}\,\mathbb{E}_u - 2\,\mathbb{F}\,\mathbb{F}_u + \mathbb{F}\,\mathbb{E}_v}{2\,(\mathbb{E}\,\mathbb{G} - \mathbb{F}^2)} = 0, & \Gamma_{11}^2 &= \frac{2\,\mathbb{E}\,\mathbb{F}_u - \mathbb{E}\,\mathbb{E}_v + \mathbb{F}\,\mathbb{E}_u}{2\,(\mathbb{E}\,\mathbb{G} - \mathbb{F}^2)} = 0, \\
\Gamma_{12}^1 &= \frac{\mathbb{G}\,\mathbb{E}_v - \mathbb{F}\,\mathbb{G}_u}{2\,(\mathbb{E}\,\mathbb{G} - \mathbb{F}^2)} = 0, & \Gamma_{12}^2 &= \frac{\mathbb{E}\,\mathbb{G}_u - \mathbb{F}\,\mathbb{E}_v}{2\,(\mathbb{E}\,\mathbb{G} - \mathbb{F}^2)} = -\tan u, \\
\Gamma_{22}^1 &= \frac{2\,\mathbb{G}\,\mathbb{F}_v - \mathbb{G}\,\mathbb{G}_u - \mathbb{F}\,\mathbb{G}_v}{2\,(\mathbb{E}\,\mathbb{G} - \mathbb{F}^2)} = \sin u \cos u, & \Gamma_{22}^2 &= \frac{\mathbb{E}\,\mathbb{G}_v - 2\,\mathbb{F}\,\mathbb{F}_v + \mathbb{F}\,\mathbb{G}_u}{2\,(\mathbb{E}\,\mathbb{G} - \mathbb{F}^2)} = 0.
\end{aligned} \tag{16}$$

2. Geodesics: Definition and Basic Properties

We have seen that there are three equivalent ways to characterize a curve γ on a surface S being "as straight as possible" curves on a curved surface,

1. Curvature of the curve at $p \in \gamma$ equals $|\kappa_n(p)|$ where $\kappa_n(p)$ is the normal curvature of S at p in the tangent direction of γ.

2. The geodesic curvature of the curve is zero, that is $\kappa_g(t) = 0$;
3. The covariant derivative of the unit tangent vector of the curve is zero along the curve, that is $\nabla_\gamma T = 0$.

Now we give a name to these "as straight as possible" curves.

DEFINITION 3. (GEODESICS) *A parametrized curve $\gamma(t)$ on the surface S is called a geodesic if $\nabla_\gamma \dot{\gamma}(t) = 0$.*

Exercise 1. In last lecture we claimed that $\nabla_\gamma w = 0$ is independent of parametrization of the curve. Is there any contradiction here?

PROPOSITION 4. *Let $\gamma(t)$ be a geodesic. Then $\|\dot{\gamma}(t)\|$ is constant.*

Proof. We have

$$0 = \nabla_\gamma \dot{\gamma} = \ddot{\gamma} - (\ddot{\gamma} \cdot N_S)\, N_S. \tag{17}$$

Thus $\ddot{\gamma} \parallel N_S$, consequently $\ddot{\gamma} \perp \dot{\gamma}$ and the conclusion follows. □

Remark 5. By this definition a "geodesic" refers to a curve, not a trace. On the other hand, if a curve $\gamma(t)$ can be re-parametrized to be a geodesic, it will be called a "pre-geodesic".

3. Geodesic equations

- [1]An arc length parametrized curve $\gamma(s) = \sigma(u(s), v(s))$ is a geodesic $\Longleftrightarrow$

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathbb{E}\,\dot{u} + \mathbb{F}\,\dot{v}) = \frac{1}{2}\,(\mathbb{E}_u\,(\dot{u})^2 + 2\,\mathbb{F}_u\,\dot{u}\,\dot{v} + \mathbb{G}_u\,(\dot{v})^2), \tag{18}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathbb{F}\,\dot{u} + \mathbb{G}\,\dot{v}) = \frac{1}{2}\,(\mathbb{E}_v\,(\dot{u})^2 + 2\,\mathbb{F}_v\,\dot{u}\,\dot{v} + \mathbb{G}_v\,(\dot{v})^2). \tag{19}$$

(18–19) are called *geodesic equations*.

Proof. (18–19) is equivalent to $\kappa = |\kappa_n|$ everywhere along the curve. We recall that

$$\kappa(s) = |\ddot{u}(s)\,\sigma_u + \ddot{v}(s)\,\sigma_v + \dot{u}(s)^2\,\sigma_{uu} + 2\,\dot{u}(s)\,\dot{v}(s)\,\sigma_{uv} + \dot{v}(s)^2\,\sigma_{vv}| \tag{20}$$

and

$$\kappa_n(s) = (\dot{u}(s)^2\,\sigma_{uu} + 2\,\dot{u}(s)\,\dot{v}(s)\,\sigma_{uv} + \dot{v}(s)^2\,\sigma_{vv}) \cdot N_S. \tag{21}$$

Thus if $\kappa(s) = |\kappa_n(s)|$, necessarily

$$(\ddot{u}(s)\,\sigma_u + \ddot{v}(s)\,\sigma_v + \dot{u}(s)^2\,\sigma_{uu} + 2\,\dot{u}(s)\,\dot{v}(s)\,\sigma_{uv} + \dot{v}(s)^2\,\sigma_{vv}) \parallel N_S \tag{22}$$

or equivalently

$$(\ddot{u}(s)\,\sigma_u + \ddot{v}(s)\,\sigma_v + \dot{u}(s)^2\,\sigma_{uu} + 2\,\dot{v}(s)\,\dot{v}(s)\,\sigma_{uv} + \dot{v}(s)^2\,\sigma_{vv}) \cdot \sigma_u = 0 \tag{23}$$

and

$$(\ddot{u}(s)\,\sigma_u + \ddot{v}(s)\,\sigma_v + \dot{u}(s)^2\,\sigma_{uu} + 2\,\dot{u}(s)\,\dot{v}(s)\,\sigma_{uv} + \dot{v}(s)^2\,\sigma_{vv}) \cdot \sigma_v = 0. \tag{24}$$

1. Theorem 9.2.1 of the textbook.

Now we prove that (23) is equivalent to (18). The proof for (24) $\Longleftrightarrow$ (19) is almost identical and omitted.

Notice that

$$\sigma_u \cdot \sigma_u = \mathbb{E}, \qquad \sigma_v \cdot \sigma_u = \mathbb{F}, \tag{25}$$

$$\sigma_{uu} \cdot \sigma_u = \left(\frac{\sigma_u \cdot \sigma_u}{2}\right)_u = \frac{1}{2}\,\mathbb{E}_u, \quad \sigma_{uv} \cdot \sigma_u = \left(\frac{\sigma_u \cdot \sigma_u}{2}\right)_v = \frac{1}{2}\,\mathbb{E}_v, \tag{26}$$

and finally

$$\sigma_{vv} \cdot \sigma_u = (\sigma_v \cdot \sigma_u)_v - \sigma_v \cdot \sigma_{vu} = \mathbb{F}_v - \frac{1}{2}\,\mathbb{G}_u. \tag{27}$$

Substituting these into (23) we obtain

$$\mathbb{E}\,\ddot{u} + \mathbb{F}\,\ddot{v} + \frac{1}{2}\,\mathbb{E}_u\,(\dot{u})^2 + \mathbb{E}_v\,\dot{u}\,\dot{v} + \mathbb{F}_v\,(\dot{v})^2 - \frac{1}{2}\,\mathbb{G}_u\,(\dot{v})^2 = 0. \tag{28}$$

On the other hand, expanding the left hand side of (18) we have

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}(\mathbb{E}\,\dot{u} + \mathbb{F}\,\dot{v}) &= \frac{\mathrm{d}\mathbb{E}(u(t), v(t))}{\mathrm{d}t}\,\dot{u} + \frac{\mathrm{d}\mathbb{F}(u(t), v(t))}{\mathrm{d}t}\,\dot{v} + \mathbb{E}\,\ddot{u} + \mathbb{F}\,\ddot{v} \\ &= \mathbb{E}_u\,(\dot{u})^2 + \mathbb{E}_v\,\dot{u}\,\dot{v} + \mathbb{F}_u\,\dot{u}\,\dot{v} + \mathbb{F}_v\,(\dot{v})^2 + \mathbb{E}\,\ddot{u} + \mathbb{F}\,\ddot{v}. \end{aligned} \tag{29}$$

The conclusion now trivially follows. $\square$

Remark 6. [2]Any local isometry between two surfaces takes the geodesics of one surface to the geodesics of the other.

Remark 7. It is useful to notice that

$$\mathbb{E}\,\dot{u} + \mathbb{F}\,\dot{v} = \dot{\gamma}(s) \cdot \sigma_u, \qquad \mathbb{F}\,\dot{u} + \mathbb{G}\,\dot{v} = \dot{\gamma}(s) \cdot \sigma_v \tag{30}$$

and the right hand side takes simpler form in matrix form:

$$\frac{1}{2}\begin{pmatrix} \dot{u} & \dot{v} \end{pmatrix}\begin{pmatrix} \mathbb{E} & \mathbb{F} \\ \mathbb{F} & \mathbb{G} \end{pmatrix}_u \begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix}, \qquad \frac{1}{2}\begin{pmatrix} \dot{u} & \dot{v} \end{pmatrix}\begin{pmatrix} \mathbb{E} & \mathbb{F} \\ \mathbb{F} & \mathbb{G} \end{pmatrix}_v \begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix}. \tag{31}$$

- [3]Alternatively, we can start from the equivalent characterization $\nabla_\gamma \dot{\gamma} = 0$ and obtain the following equations.

$$\ddot{u} + \Gamma_{11}^1\,(\dot{u})^2 + 2\,\Gamma_{12}^1\,\dot{u}\,\dot{v} + \Gamma_{22}^1\,(\dot{v})^2 = 0, \tag{32}$$

$$\ddot{v} + \Gamma_{11}^2\,(\dot{u})^2 + 2\,\Gamma_{12}^2\,\dot{u}\,\dot{v} + \Gamma_{22}^2\,(\dot{v})^2 = 0. \tag{33}$$

Remark 8. (32) and (33) take simple matrix forms:

$$\ddot{u} + \begin{pmatrix} \dot{u} & \dot{v} \end{pmatrix}\begin{pmatrix} \Gamma_{11}^1 & \Gamma_{12}^1 \\ \Gamma_{12}^1 & \Gamma_{22}^1 \end{pmatrix}\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = 0, \tag{34}$$

2. Corollary 9.2.7 of the textbook.

3. Proposition 9.2.3 of the textbook.

and

$$\ddot{v} + (\,\dot{u}\ \ \dot{v}\,)\begin{pmatrix} \Gamma_{11}^2 & \Gamma_{12}^2 \\ \Gamma_{12}^2 & \Gamma_{22}^2 \end{pmatrix}\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = 0. \tag{35}$$

The pattern would be crystal clear if we replace u by u_1 and v by u_2.

- We note that in deriving (18–19), we have assumed arc length parametrization, while in deriving (32–33) this assumption is not needed. Nevertheless, (18–19) is equivalent to (32–33). In particular, we have the following.

PROPOSITION 9. *If $\gamma(t) = \sigma(u(t), v(t))$ satisfy (18–19), then $\|\dot{\gamma}(t)\|$ is a constant.*

Exercise 2. Prove Proposition 9 in two ways.

i. Directly calculate $\frac{\mathrm{d}}{\mathrm{d}t}\|\dot{\gamma}(t)\|$;

ii. Show that (18–19) and (32–33) are equivalent.

4. Examples (Clairaut's Theorem)

Example 10. [4]We consider a surface of revolution

$$\sigma(u, v) = (f(u)\cos v,\, f(u)\sin v, u). \tag{36}$$

We have

$$\sigma_u = (f'(u)\cos v,\, f'(u)\sin v, 1), \qquad \sigma_v = (-f(u)\sin v,\, f(u)\cos v, 0) \tag{37}$$

which leads to

$$\mathbb{E} = 1 + f'(u)^2, \qquad \mathbb{F} = 0, \qquad \mathbb{G} = f(u)^2. \tag{38}$$

Thus (18–19) becomes

$$\frac{\mathrm{d}}{\mathrm{d}s}((1 + f'(u)^2)\,\dot{u}) = f'(u)\,f''(u)\,(\dot{u})^2 + f(u)\,f'(u)\,(\dot{v})^2, \tag{39}$$

$$\frac{\mathrm{d}}{\mathrm{d}s}(f(u)^2\,\dot{v}) = 0. \tag{40}$$

We see that $f(u)^2\,\dot{v} = C$ is a constant. The first equation simplifies to

$$(1 + f'(u)^2)\,\ddot{u} = f(u)\,f'(u)\,(\dot{v})^2. \tag{41}$$

We see that

- $v = $ constant are geodesics;
- $u = u_0$ where $f'(u_0) = 0$ are geodesics.
- We see that

$$\cos\angle(\sigma_u, \dot{\gamma}) = \frac{\sqrt{1 + f'(u)^2}\,\dot{u}}{\sqrt{(1 + f'(u)^2)\,(\dot{u})^2 + f(u)^2\,(\dot{v})^2}}. \tag{42}$$

Exercise 3. Prove (42) though calculation using the first fundamental form, without explicit application of (37).

4. Proposition 9.3.1, 9.3.2 of the textbook.

which gives

$$\sin^2\angle(\sigma_u,\dot\gamma)=\frac{f(u)^2\,(\dot v)^2}{(1+f'(u)^2)\,(\dot u)^2+f(u)^2\,(\dot v)^2}. \tag{43}$$

Now notice that, as $\gamma(t)$ is geodesic, $(1+f'(u)^2)\,(\dot u)^2+f(u)^2\,(\dot v)^2=\|\dot\gamma(t)\|^2=\text{constant}$. Consequently we have

$$f(u)^2\sin^2\angle(\sigma_u,\dot\gamma)=\text{constant}. \tag{44}$$

Finally, if this constant if not zero, by continuity and the assumption that $f(u)\neq 0$, we must have

$$f(u)\sin\angle(\sigma_u,\dot\gamma)=\text{constant}. \tag{45}$$

Thus if $\gamma(t)$ is a geodesic, then there is a constant c_0 such that

$$\sin\angle(\sigma_u,\dot\gamma)=\frac{c_0}{f(u)}. \tag{46}$$

On the other hand, if $\gamma(t)$ is a curve on the surface satisfying (46), then its arc length parametrization $\Gamma(s)$ still satisfies (46), and it turns out that $\Gamma(s)$ is a geodesic.

Exercise 4. Prove that (46)$\Longleftrightarrow$(40). Next prove that if $\|\dot\gamma(t)\|=\text{constant}$ and $\gamma(t)$ satisfies one of (18–19), then it must satisfy the other. This means that (39) is also satisfied.

In summary, we have

THEOREM 11. (CLAIRAUT'S THEOREM) *Geodesics on a surface of revolution $\sigma(u,v)=(f(u)\cos v, f(u)\sin v, u)$ is characterized by (46).*

Example 12. Let S be the unit sphere. This of course is a surface of revolution. We re-write it as

$$(f(u)\cos v, f(u)\sin v, u) \tag{47}$$

where $f(u)=\cos(\arcsin u)=\sqrt{1-u^2}$. By Example 10 we see that for any geodesic $\gamma(s)=(f(u(s))\cos v(s), f(u(s))\sin v(s), u(s))$, parametrized by arc length, there holds

$$(1-u^2)\,\dot v=f(u)^2\,\dot v=\text{constant}=:c_0. \tag{48}$$

On the other hand, as $\|\dot\gamma(s)\|=1$ we have

$$1=f'(u)^2\,(\dot u)^2+f(u)^2\,(\dot v)^2+(\dot u)^2=\frac{(\dot u)^2}{1-u^2}+(1-u^2)\,(\dot v)^2. \tag{49}$$

Therefore

$$1-c_0^2=u^2+(\dot u)^2. \tag{50}$$

Now we have (we simply write u for $u(s)$)

$$\gamma(s)=\left(\sqrt{1-u^2}\cos v, \sqrt{1-u^2}\sin v, u\right) \tag{51}$$

and

$$\dot\gamma(s)=\left(\frac{-u\,\dot u}{\sqrt{1-u^2}}\cos v-\sqrt{1-u^2}\sin v\,\dot v, \frac{-u\,\dot u}{\sqrt{1-u^2}}\sin v+\sqrt{1-u^2}\cos v\,\dot v, \dot u\right). \tag{52}$$

Consequently

$$\gamma(s)\times\dot{\gamma}(s)=\left(\frac{\dot{u}\sin v-c_0\,u\cos v}{\sqrt{1-u^2}},-\frac{\dot{u}\cos v+c_0\,u\sin v}{\sqrt{1-u^2}},c_0\right) \tag{53}$$

Now calculate (using (48) and (50) whenever applicable)

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\dot{u}\sin v-c_0\,u\cos v}{\sqrt{1-u^2}}\right) &= \frac{\ddot{u}\sin v+\dot{u}\cos v\,\dot{v}-c_0\,\dot{u}\cos v+c_0\,u\sin v\,\dot{v}}{\sqrt{1-u^2}}\\ &\quad+\frac{u\,\dot{u}\,(\dot{u}\sin v-c_0\,u\cos v)}{\sqrt{1-u^2}^3}\\ &= \frac{\ddot{u}\,(1-u^2)\sin v+(c_0^2+(\dot{u})^2)u\sin v}{\sqrt{1-u^2}^3}\\ &= \frac{\ddot{u}\,(1-u^2)\sin v+(1-u^2)\,u\sin v}{\sqrt{1-u^2}^3}.\end{aligned} \tag{54}$$

Finally notice that differentiating (50) gives

$$(\ddot{u}+u)\,\dot{u}=0. \tag{55}$$

Thus if $\dot{u}\neq 0$, there holds $\ddot{u}=-u$ and consequently $\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\dot{u}\sin v-c_0\,u\cos v}{\sqrt{1-u^2}}\right)=0$. Similarly $\frac{\mathrm{d}}{\mathrm{d}s}\left(-\frac{\dot{u}\cos v+c_0\,u\sin v}{\sqrt{1-u^2}}\right)=0$ and $\gamma(s)\times\dot{\gamma}(s)$ is a constant vector. This implies $\gamma(s)$ lies in a plane passing the origin and must be part of a big circle.

Exercise 5. Rigorously prove this last claim: Let $\gamma(s)$ be a curve on the unit sphere parametrized by arc length. Assume that $\gamma(s)\times\dot{\gamma}(s)$ is a constant vector. Then $\gamma(s)$ lies in a plane passing the origin.

Exercise 6. What if $\dot{u}=0$?

5. Geodesics as shortest paths

- On the flat plane, the shortest path connecting any two points is the one that is part of a geodesic, which is a straight line.
- Set up. Let $p_1,p_2\in S$ and let γ_0 be the shortest path connecting p_1,p_2. Parametrize γ_0 by arc length $\sigma(u_0(s),v_0(s))$. Set $\gamma(s_1)=p_1$, $\gamma(s_2)=p_2$. Now let $(u_1(s),v_1(s))$ be an arbitrary vector field along $(u_0(s),\,v_0(s))$ in $\mathbb{R}^2$ and $\lambda(s)\colon\mathbb{R}\mapsto\mathbb{R}$ be such that $\lambda(s_1)=\lambda(s_2)=0$. Let $\tau\in\mathbb{R}$ and denote by γ_τ the curve $\sigma(u_0+\tau\,\lambda\,u_1,\,v_0+\tau\lambda\,v_1)$. Finally denote

$$L(\tau)=\int_{s_1}^{s_2}\left\|\frac{\mathrm{d}}{\mathrm{d}s}\sigma(u_0(s)+\tau\,\lambda(s)\,u_1(s),v_0(s)+\tau\,\lambda(s)\,v_1(s))\right\|\mathrm{d}s. \tag{56}$$

- Calculus of variations. We have

$$\begin{aligned}L(\tau) &= \int_{s_1}^{s_2}\left\|\frac{\mathrm{d}}{\mathrm{d}s}(u_0+\tau\,\lambda\,u_1)\,\sigma_u+\frac{\mathrm{d}}{\mathrm{d}s}(v_0+\tau\,\lambda\,v_1)\,\sigma_v\right\|\mathrm{d}s\\ &= \int_{s_1}^{s_2}\left[\left(\frac{\mathrm{d}}{\mathrm{d}s}(u_0+\tau\,\lambda\,u_1)\,\sigma_u+\frac{\mathrm{d}}{\mathrm{d}s}(v_0+\tau\,\lambda\,v_1)\,\sigma_v\right)\cdot\left(\frac{\mathrm{d}}{\mathrm{d}s}(u_0+\tau\,\lambda\,u_1)\,\sigma_u+\right.\right.\\ &\quad\left.\left.\frac{\mathrm{d}}{\mathrm{d}s}(v_0+\tau\,\lambda\,v_1)\,\sigma_v\right)\right]^{1/2}\mathrm{d}s.\end{aligned} \tag{57}$$

Here it is crucial to realize that σ_u, σ_v are evaluated at $(u_0 + \tau\,\lambda\,u_1, v_0 + \tau\,\lambda\,v_1)$. In particular, they are dependent on τ.

Now we calculate

$$\begin{aligned} L'(0) &= \int_{s_1}^{s_2} (\mathbb{E}_0\,\dot{u}_0^2 + 2\,\mathbb{F}_0\,\dot{u}_0\,\dot{v}_0 + \mathbb{G}_0\,\dot{v}_0^2)^{-1/2} \left[(\dot{u}_0\,\sigma_u + \dot{v}_0\,\sigma_v) \cdot \left((\lambda\,\dot{u}_1)\,\sigma_u + (\lambda\,\dot{v}_1)\,\sigma_v + \right.\right. \\ &\quad \left.\left. \dot{u}_0\,(\lambda\,u_1\,\sigma_{uu} + \lambda\,v_1\,\sigma_{uv}) + \dot{v}_0\,(\lambda\,u_1\,\sigma_{vu} + \lambda\,v_1\,\sigma_{vv}) \right) \right] \mathrm{d}s \\ &= \int_{s_1}^{s_2} (\lambda\,\dot{u}_1)\,[(\dot{u}_0\,\sigma_u + \dot{v}_0\,\sigma_v) \cdot \sigma_u] + (\lambda\,\dot{v}_1)\,[(\dot{u}_0\,\sigma_u + \dot{v}_0\,\sigma_v) \cdot \sigma_v]\,\mathrm{d}s \\ &\quad + \int_{s_1}^{s_2} (\lambda\,u_1)\,[(\dot{u}_0\,\sigma_{uu} + \dot{v}_0\,\sigma_{uv}) \cdot (\dot{u}_0\,\sigma_u + \dot{v}_0\,\sigma_v)]\,\mathrm{d}s \\ &\quad + \int_{s_1}^{s_2} (\lambda\,v_1)\,[(\dot{u}_0\,\sigma_{uv} + \dot{v}_0\,\sigma_{vv}) \cdot (\dot{u}_0\,\sigma_u + \dot{v}_0\,\sigma_v)]\,\mathrm{d}s. \end{aligned} \tag{58}$$

Note that in the above we have used the fact that $\mathbb{E}_0\ \dot{u}_0^2 + 2\ \mathbb{F}_0\ \dot{u}_0\ \dot{v}_0 + \mathbb{G}_0\ \dot{v}_0^2 = \|\dot{\gamma}(s)\|^2 = 1$. Also note that in (58) $\sigma_u, \ldots, \sigma_{vv}$ are all evaluated at (u_0, v_0) now.

Next we integrate the first integral in (58) by parts and collect all the u_1 terms together, and all the v_1 terms together.

$$\begin{aligned} L'(0) &= -\int_{s_1}^{s_2} (\lambda\,u_1) \left[\frac{\mathrm{d}}{\mathrm{d}s}((\dot{u}_0\,\sigma_u + \dot{v}_0\,\sigma_v) \cdot \sigma_u) - (\dot{u}_0\,\sigma_{uu} + \dot{v}_0\,\sigma_{uv}) \cdot (\dot{u}_0\,\sigma_u + \dot{v}_0\,\sigma_v) \right] \mathrm{d}s \\ &\quad - \int_{s_1}^{s_2} (\lambda\,v_1) \left[\frac{\mathrm{d}}{\mathrm{d}s}((\dot{u}_0\,\sigma_u + \dot{v}_0\,\sigma_v) \cdot \sigma_v) - (\dot{u}_0\,\sigma_{uv} + \dot{v}_0\,\sigma_{vv}) \cdot (\dot{u}_0\,\sigma_u + \dot{v}_0\,\sigma_v) \right] \mathrm{d}s. \end{aligned}$$

Due to the arbitrariness of u_1, v_1, we conclude

$$\frac{\mathrm{d}}{\mathrm{d}s}((\dot{u}_0\,\sigma_u + \dot{v}_0\,\sigma_v) \cdot \sigma_u) = (\dot{u}_0\,\sigma_{uu} + \dot{v}_0\,\sigma_{uv}) \cdot (\dot{u}_0\,\sigma_u + \dot{v}_0\,\sigma_v), \tag{59}$$

$$\frac{\mathrm{d}}{\mathrm{d}s}((\dot{u}_0\,\sigma_u + \dot{v}_0\,\sigma_v) \cdot \sigma_v) = (\dot{u}_0\,\sigma_{uv} + \dot{v}_0\,\sigma_{vv}) \cdot (\dot{u}_0\,\sigma_u + \dot{v}_0\,\sigma_v). \tag{60}$$

Simple calculation now gives

$$\frac{\mathrm{d}}{\mathrm{d}s}(\mathbb{E}\,\dot{u}_0 + \mathbb{F}\,\dot{v}_0) = \frac{1}{2}\,(\mathbb{E}_u\,(\dot{u}_0)^2 + 2\,\mathbb{F}_u\,\dot{u}_0\,\dot{v}_0 + \mathbb{G}_u\,(\dot{v}_0)^2), \tag{61}$$

$$\frac{\mathrm{d}}{\mathrm{d}s}(\mathbb{F}\,\dot{u}_0 + \mathbb{G}\,\dot{v}_0) = \frac{1}{2}\,(\mathbb{E}_v\,(\dot{u}_0)^2 + 2\,\mathbb{F}_v\,\dot{u}_0\,\dot{v}_0 + \mathbb{G}_v\,(\dot{v}_0)^2). \tag{62}$$

Exercise 7. Derive (61–62) from (59–60).

Remark 13. Note that a shortest path must be a geodesic but a geodesic does not necessarily give shortest path.

Remark 14. [5]It can be show that if $\gamma(s) = \sigma(u(s), v(s))$ satisfy the geodesic equations, then $\|\dot{\gamma}(s)\|$ is a constant. On the other hand, an arbitrary curve $\gamma(t) = \sigma(u(t), v(t))$, not necessarily with constant speed, has the same trace as a geodesic if and only if $\ddot{\gamma} \cdot (N_S \times \dot{\gamma}) = 0$ along γ.

5. Proposition 9.1.2 and Exercise 9.1.2 of the textbook.