Math 348 Differential Geometry of Curves and Surfaces

Lecture 15 Parallel Transport

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Please do not hesitate to interrupt me if you have a question.

Motivation, Definitions, and Properties

Motivation

- Parallel vectors in Euclidean spaces;
- Difficulty for curved surfaces: How to compare two vectors belonging to two different tangent spaces.
- Treating them as vectors in the ambient Euclidean space: Not working.
- Idea: "Move" a vector from one point to the other, without changing direction, and then compare.

Tangent vector field of a surface along a curve

- $\mathcal{C} = \gamma(t)$: A curve on S;
- $w : \mathcal{C} \mapsto \mathbb{R}^3$, $w(p) \in T_pS$ for every $p \in \mathcal{C}$;
- w is called a "tangent vector field along $\mathcal{C}$".
- Can similarly define a "tangent vector field" of S.

Covariant derivative

- **Idea.** "Horizontal" change of a vector field.
- **Definition.**

$$\nabla_\gamma w = \dot{w} - (\dot{w} \cdot N_S) N_S$$

  where N_S is the unit normal of the surface.
- **Parallel.** A vector field w is parallel along $\gamma(t)$: $\nabla_\gamma w = 0$ at every point along the curve.
- **Properties.**
  1. $\nabla_\gamma w = 0 \Leftrightarrow \dot{w} \perp T_p S \Leftrightarrow \dot{w}(t) \parallel N_S(p)$.
  2. $\nabla_\gamma w = 0$ is independent of parametrization of the curve.

Parallel vectors

$$\boxed{\text{A vector field } w \text{ is parallel along } \gamma(t)\text{: } \nabla_\gamma w = 0.}$$

Example

1. S: The xy plane; $\gamma(t) = (u(t), v(t), 0)$, $w(t) = \dot{\gamma}(t)$;
2. S: The unit cylinder; $\gamma(t) = (\cos u(t), \sin u(t), v(t))$, $w(t) = \dot{\gamma}(t)$;
3. S: The unit sphere; $\gamma(t)$: Fixed latitude.
   - $w(t) = \dot{\gamma}(t)$;
   - $w(t)$ is the unit vector "pointing north".

Properties

1. There is exactly one vector field along the curve parallel to a fixed tangent vector at some fixed point p_0 on the curve;
2. If two parametrizations of γ has the same $\dot{\gamma}$ at a point p, then $\nabla_{\gamma} w$ are the same at p;
3. Thus the covariant derivative is a function of the point p and the tangent direction $\dot{\gamma} \in T_p S$.
4. Let $\gamma(t)$ be a curve on S with $T(t)$ its unit tangent vector. Then the following are equivalent (along the curve).
   4.1 $\kappa = |\kappa_n|$;
   4.2 $\kappa_g = 0$;
   4.3 $\nabla_{\gamma} T = 0$;

Parallel transport map

$$\boxed{p, q \in S;\ w_0 \in T_pS, w_1 \in T_qS.}$$

- It does not make sense to say "w_0 and w_1 are parallel"; Example: Unit sphere.
- Have to say w_0 and w_1 are parallel along γ, where γ is a curve in S connecting p, q;
- **Parallel transport map.** $w(t)$: Unique vector field along γ parallel to w_0. The parallel transport of w_0 to q along γ is w at q.
- **Notation.**

$$\Pi_\gamma^{pq} : T_pS \mapsto T_qS.$$

- Π_γ^{pq} is an isometry.

Calculation and Christoffel Symbols

Christoffel symbols

$$\boxed{\text{Given: } \mathbb{E}, \mathbb{F}, \mathbb{G}, \mathbb{L}, \mathbb{M}, \mathbb{N}}$$

$$\sigma_{uu} = \Gamma^1_{11}\sigma_u + \Gamma^2_{11}\sigma_v + \mathbb{L}N \quad (1)$$

$$\sigma_{uv} = \Gamma^1_{12}\sigma_u + \Gamma^2_{12}\sigma_v + \mathbb{M}N \quad (2)$$

$$\sigma_{vv} = \Gamma^1_{22}\sigma_u + \Gamma^2_{22}\sigma_v + \mathbb{N}N \quad (3)$$

- See textbook or lecture notes for formulas for Γ^k_{ij};
- Spot the pattern;

Parallel transport equation

$$\boxed{w(t) = \alpha(t)\sigma_u + \beta(t)\sigma_v \text{ along } \gamma(t) = \sigma(u(t), v(t)).}$$

$w(t)$ is parallel along $\gamma(t)$ if and only if

$$\begin{aligned} \dot{\alpha} + (\Gamma^1_{11}\dot{u} + \Gamma^1_{12}\dot{v})\alpha + (\Gamma^1_{12}\dot{u} + \Gamma^1_{22}\dot{v})\beta &= 0 & (4) \\ \dot{\beta} + (\Gamma^2_{11}\dot{u} + \Gamma^2_{12}\dot{v})\alpha + (\Gamma^2_{12}\dot{u} + \Gamma^2_{22}\dot{v})\beta &= 0. & (5) \end{aligned}$$

Examples

$\boxed{\text{Conditions for } w(t) = \alpha(t)\sigma_u + \beta(t)\sigma_v \text{ to be parallel along } \gamma.}$

1. Plane;
2. Cylinder;
3. Sphere.

Looking Back and Forward

Summary

Required: §7.4, §9.1–9.4; Optional: §9.5.

- **Properties.**
  1. Covariant derivative: $\nabla_\gamma w = \dot{w} - (\dot{w} \cdot N_S) N_S$;
  2. Parallel vector field along γ: $\nabla_\gamma w = 0$;
  3. Parallel transport map;
  4. Christoffel symbols.

$$\begin{aligned} \sigma_{uu} &= \Gamma^1_{11}\sigma_u + \Gamma^2_{11}\sigma_v + \mathbb{L}N && (6) \\ \sigma_{uv} &= \Gamma^1_{12}\sigma_u + \Gamma^2_{12}\sigma_v + \mathbb{M}N && (7) \\ \sigma_{vv} &= \Gamma^1_{22}\sigma_u + \Gamma^2_{22}\sigma_v + \mathbb{N}N && (8) \end{aligned}$$

  5. Parallel transport equations.

$$\begin{aligned} \dot{\alpha} + (\Gamma^1_{11}\dot{u} + \Gamma^1_{12}\dot{v})\alpha + (\Gamma^1_{12}\dot{u} + \Gamma^1_{22}\dot{v})\beta &= 0 && (9) \\ \dot{\beta} + (\Gamma^2_{11}\dot{u} + \Gamma^2_{12}\dot{v})\alpha + (\Gamma^2_{12}\dot{u} + \Gamma^2_{22}\dot{v})\beta &= 0. && (10) \end{aligned}$$

See you Thursday!

$\boxed{\text{Geodesics.}}$

1. Motivation and definition;
2. Geodesic equations;
3. Shortest path.