Lectures 15: Parallel Transport

Disclaimer. As we have a textbook, this lecture note is for guidance and supplement only. It should not be relied on when preparing for exams.

In this lecture we study the how two vectors at different points on a surface S can be said to be "parallel" to each other.

The required textbook sections are §7.4, §9.1–9.4. The optional sections are §9.5

I try my best to make the examples in this note different from examples in the textbook. Please read the textbook carefully and try your hands on the exercises. During this please don't hesitate to contact me if you have any questions.

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1. Motivation and definitions

- In classical geometry, one of the most important ideas is parallelism. Two vectors are "parallel" if we can "move" one of them, without changing its direction, to coincide with the other.
- However on a curved surface "without changing its direction" becomes problematic. For example consider the following situation:

  Consider the unit sphere. Let B be the north pole and A, B be two points on the equator. Then clearly v_B should be the tangent vector at B that is "parallel" to v_A at A. Similarly $v_C \parallel v_B$, $\tilde{v}_A \parallel v_C$. However it is clear that $v_A \nparallel \tilde{v}_A$.

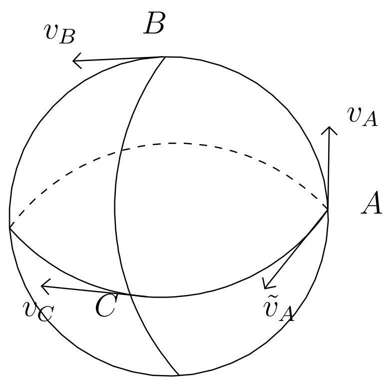

Figure 1. $v_A \parallel v_B, v_B \parallel v_C, v_C \parallel \tilde{v}_A$.

- One possible way to fix this is the following.
  - Recall that in Euclidean geometry, to guarantee "without changing its direction", we draw a straight line connecting the two base points, and then measure the angles between the vectors and this line. As straight line segments are shortest paths between the two base points, this approach works for curved surfaces.
- Another possibility is the following more intuitive appraoch.
- Instead of comparing vectors at two different points, we consider the following situation.

  Let $\mathcal{C}$ be a curve on a surface S. Let w be a tangent vector field along $\mathcal{C}$, that is $w: S \mapsto \mathbb{R}^3$ such that $w(p) \in T_p S$ for every $p \in \gamma$.

Remark 1. As soon as $\mathcal{C}$ is parametrized by some $\gamma(t)$, we can form the composite function $w(\gamma(t))$. When no confusion should arise, we abuse notation a bit and simply write $w(t)$.

Remark 2. It is easy to see how "a tangent vector field on S" should be defined.

Exercise 1. Give a reasonable definition to a "tangent vector field on S".

We would like to give definition to "w does not change direction along $\mathcal{C}$".

- One reasonable definition is the following.
  - Covariant derivative.

    In the above setting, the covariant derivative of w along $\gamma(t)$ is given by the tangential component of $\dot{w}$:

    $$\nabla_\gamma w = \dot{w} - (\dot{w} \cdot N_S)\, N_S \tag{1}$$

    where N_S is the unit normal of the surface.
  - Then we say w to be parallel along γ if $\nabla_\gamma w = 0$ at every point of γ.

Remark 3. Clearly,

$$\nabla_\gamma w = 0 \Longleftrightarrow \dot{w} \perp T_{\gamma(t)} S \Longleftrightarrow \dot{w}(t) \parallel N_S(\gamma(t)). \tag{2}$$

Example 4. Let S be the x-y plane. Let $\gamma(t) = (u(t), v(t), 0)$. Let $w(t) := \dot{\gamma}(t) = (\dot{u}(t), \dot{v}(t), 0)$ and

$$\nabla_\gamma w(t) = \ddot{\gamma}(t) - [\ddot{\gamma}(t) \cdot N_S(t)] N_S(t) = (\ddot{u}(t), \ddot{v}(t), 0) \tag{3}$$

as $N_S(t) = (0, 0, 1)$ for all t. Consequently $\dot{\gamma}(t)$ is parallel along γ if and only if $\ddot{u}(t) = \ddot{v}(t) = 0$, that is $u = a_1 t + a_0, v = b_1 t + b_0$.

Thus a plane curve is "straight" when it is a straight line.

Remark 5. Thus we see that "parallel transport" is slightly different from our intuitive idea of "parallel".

Example 6. Let S be the cylinder $\sigma(u, v) = (\cos u, \sin u, v)$. Let $\gamma(t) = (\cos u(t), \sin u(t), v(t))$. Let

$$w(t) := \dot{\gamma}(t) = ((-\sin u(t))\, \dot{u}(t), (\cos u(t))\, \dot{u}(t), \dot{v}(t)) \tag{4}$$

and

$$\dot{w}(t) = ((-\cos u)\, \dot{u}^2 - (\sin u)\, \ddot{u}, (-\sin u)\, \dot{u}^2 + (\cos u)\, \ddot{u}, \ddot{v}). \tag{5}$$

On the other hand, we have

$$\sigma_u = (-\sin u, \cos u, 0), \qquad \sigma_v = (0, 0, 1), \tag{6}$$

therefore

$$N_S = \frac{\sigma_u \times \sigma_v}{\|\sigma_u \times \sigma_v\|} = (\cos u, \sin u, 0). \tag{7}$$

Thus we can calculate

$$\nabla_\gamma w(t) = (-(\sin u)\, \ddot{u}, (\cos u)\, \ddot{u}, \ddot{v}). \tag{8}$$

Therefore $\nabla_\gamma w(t) = 0 \Longleftrightarrow$

$$-(\sin u)\, \ddot{u} = 0, \qquad (\cos u)\, \ddot{u} = 0, \qquad \ddot{v} = 0. \tag{9}$$

This is equivalent to $\ddot{u} = 0, \ddot{v} = 0$.

Thus a cylindrical curve is "straight" when it is of the form $(\cos u(t), \sin u(t), v(t))$ where $(u(t), v(t))$ is a straight line in the plane.

Example 7. Let S be the unit sphere given by $\sigma(u,v) = (\cos u \cos v, \cos u \sin v, \sin u)$. We consider $\gamma(t) = (\cos u_0 \cos t, \cos u_0 \sin t, \sin u_0)$. Let $w(t) = \dot{\gamma}(t)$ be the tangent vector. We have

$$\dot{w}(t) = (-\cos u_0 \cos t, -\cos u_0 \sin t, 0). \tag{10}$$

On the other hand we have

$$N_S(u,v) = (\cos u \cos v, \cos u \sin v, \sin u). \tag{11}$$

Therefore

$$\nabla_\gamma w(t) = \frac{\sin 2\, u_0}{2} (-\sin u_0 \cos t, -\sin u_0 \sin t, \cos u_0). \tag{12}$$

We see that it is zero only if $u_0 = 0$, that is γ is part of a big circle.

Exercise 2. What about $u_0 = \pi/2$?

Example 8. Of course we should not restrict ourselves to the tangent of the curve. We take the setting of Example 7 and let $w(t) := (-\sin u_0 \cos t, -\sin u_0 \sin t, \cos u_0)$ be the unit tangent vector at $\gamma(t)$ "pointing north".

We have

$$\dot{w}(t) = (\sin u_0 \sin t, -\sin u_0 \cos t, 0). \tag{13}$$

Again

$$N_S(u,v) = (\cos u \cos v, \cos u \sin v, \sin u). \tag{14}$$

Therefore

$$\nabla_\gamma w(t) = \sin u_0 (\sin t, -\cos t, 0). \tag{15}$$

Again $w(t)$ is parallel along γ if and only if $u_0 = 0$, that is γ is part of a big circle.

Exercise 3. Study $\nabla_\gamma w(t)$ for $w(t) = \dot{\gamma}(t)$ for an arbitrary spherical curve.

2. Properties

Remark 9. [1]Let γ be a curve on a surface S and let w_0 be a tangent vector of S at the point $\gamma(t_0)$. Then there is exactly one tangent vector field w that is parallel along γ and is such that $w(t_0) = w_0$.

LEMMA 10. *Let $\gamma(t)$ be a curve on S. Let w be a tangent vector field along γ. Then the condition $\nabla_\gamma w = 0$ is independent of the parametrization of γ.*

Proof. Let $\gamma(t)$, $\tilde{\gamma}(\tilde{t})$ be two different parametrizations of $\gamma(t)$. Then there is a function $\tilde{T}(t)$ such that $\tilde{\gamma}(\tilde{t}) = \gamma(\tilde{T}(t))$. Since w is a vector field along γ, we have

$$\tilde{w}(\tilde{t}) = w(\tilde{T}(t)). \tag{16}$$

Consequently

$$\nabla_{\tilde{\gamma}} \tilde{w} = \dot{w}\, \dot{\tilde{T}} - \left(\dot{w}\, \dot{\tilde{T}} \cdot N_S \right) N_S = \dot{\tilde{T}} \nabla_\gamma w. \tag{17}$$

1. Corollary 7.4.6 of the textbook.

Therefore $\nabla_\gamma w = 0 \Longleftrightarrow \nabla_{\tilde{\gamma}} \tilde{w} = 0$. □

Remark 11. Note that $\nabla_\gamma w \neq \nabla_{\tilde{\gamma}} \tilde{w}$.

Exercise 4. Is $\nabla_\gamma w$ independent of the parametrization of γ?

Remark 12. Lemma 10 justifies the notation ∇_γ where parametrization is not involved. The situation can be further simplified by the following lemma, which says that covariant derivative is simply "directional derivative" on surfaces.

LEMMA 13. *Let $\gamma, \tilde{\gamma}$ be two curves on S that are tangent at $p \in S$. Let w be a tangent vector field of S, that is $w: S \mapsto \mathbb{R}^3$ with $w(p) \in T_p S$. Let γ, $\tilde{\gamma}$ be parametrized by $\gamma(t)$, $\tilde{\gamma}(\tilde{t})$ with $p = \gamma(t_0) = \tilde{\gamma}(\tilde{t}_0)$ and furthermore $\dot{\gamma}(t_0) = \dot{\tilde{\gamma}}(\tilde{t}_0)$. Then $\nabla_\gamma w = \nabla_{\tilde{\gamma}} w$ at p.*

Proof. We have

$$\frac{\mathrm{d}w}{\mathrm{d}t}(t_0) = D_p w(\dot{\gamma}(t_0)) = D_p w(\dot{\tilde{x}}(\tilde{t}_0)) = \frac{\mathrm{d}w}{\mathrm{d}\tilde{t}}(\tilde{t}_0). \tag{18}$$

Consequently

$$\nabla_\gamma w = \frac{\mathrm{d}w}{\mathrm{d}t}(t_0) - \left[\frac{\mathrm{d}w}{\mathrm{d}t}(t_0) \cdot N_S\right] N_S = \frac{\mathrm{d}w}{\mathrm{d}\tilde{t}}(\tilde{t}_0) - \left[\frac{\mathrm{d}w}{\mathrm{d}\tilde{t}}(\tilde{t}_0) \cdot N_S\right] N_S = \nabla_{\tilde{\gamma}} w, \tag{19}$$

exactly what we need to prove. □

LEMMA 14. *Let $\gamma(t)$ be a curve on S. Then the following are equivalent.*

i. Along γ there holds $\kappa(t) = |\kappa_n|$;

ii. Along γ there holds $\kappa_g = 0$;

iii. $T(t)$, the unit tangent vector to γ, is parallel along γ.

Remark 15. Thus the three seemingly different ways to characterize "as straight as possible" curves on a curved surface,

1. $\kappa(t) = |\kappa_n(\gamma(t))|$,
2. $\kappa_g(t) = 0$;
3. $\nabla_\gamma T(t) = 0$,

are all equivalent.

Proof. Thanks to Lemma 10, we can take $x(s)$ to be the arc length parametrization of γ. Then recall that by definition of κ_n, κ_g we have

$$\ddot{\gamma}(s) = \kappa_n N_S + \kappa_g (T \times N_S). \tag{20}$$

Consequently

$$\nabla_\gamma T = \ddot{\gamma}(s) - \kappa_n N_S, \tag{21}$$

and the conclusion follows. □

Exercise 5. There is a minor gap in the above argument. Can you fix it?

3. Calculation of the covariant derivative and Christoffel symbols

- How to calculate covariant derivative on an abstract surface, with only the two fundamental forms given?
- Set up. Let S be a surface parametrized by the patch $\sigma(u,v)$. Let $\gamma(t)=\sigma(u(t),v(t))$ and $w=w(u,v)$ be a tangent vector field along γ. Therefore there are $\alpha(t),\beta(t)$ such that $w=\alpha\,\sigma_u+\beta\,\sigma_v$.
- Now we calculate

$$\begin{aligned}\dot{w} &= \dot{\alpha}\,\sigma_u+\dot{\beta}\,\sigma_v+\alpha\,[\sigma_{uu}\,\dot{u}+\sigma_{uv}\,\dot{v}]+\beta\,[\sigma_{vu}\,\dot{u}+\sigma_{vv}\,\dot{v}]\\ &= \dot{\alpha}\,\sigma_u+\dot{\beta}\,\sigma_v+(\alpha\,\dot{u})\,\sigma_{uu}+(\alpha\,\dot{v}+\beta\,\dot{u})\,\sigma_{uv}+(\beta\,\dot{v})\,\sigma_{vv}.\end{aligned} \tag{22}$$

  Therefore

$$\begin{aligned}\nabla_\gamma w &= \dot{w}-(\dot{w}\cdot N_S)\,N_S\\ &= \dot{\alpha}\,\sigma_u+\dot{\beta}\,\sigma_v\\ &\quad+(\alpha\,\dot{u})\,(\sigma_{uu}-(\sigma_{uu}\cdot N_S)\,N_S)\\ &\quad+(\alpha\,\dot{v}+\beta\,\dot{u})\,(\sigma_{uv}-(\sigma_{uv}\cdot N_S)\,N_S)\\ &\quad+(\beta\,\dot{v})\,(\sigma_{vv}-(\sigma_{vv}\cdot N_S)\,N_S).\end{aligned} \tag{23}$$

- To understand this formula we introduce Christoffel symbols Γ_{ij}^k and the related Gauss equations.

PROPOSITION 16. (GAUSS EQUATIONS) *Let $\sigma(u,v)$ be a surface patch with first and second fundamental forms*

$$\mathbb{E}\,\mathrm{d}u^2+2\,\mathbb{F}\,\mathrm{d}u\,\mathrm{d}v+\mathbb{G}\,\mathrm{d}v^2 \text{ and } \mathbb{L}\,\mathrm{d}u^2+2\,\mathbb{M}\,\mathrm{d}u\,\mathrm{d}v+\mathbb{N}\,\mathrm{d}v^2. \tag{24}$$

Then

$$\sigma_{uu} = \Gamma_{11}^1\,\sigma_u+\Gamma_{11}^2\,\sigma_v+\mathbb{L}\,N, \tag{25}$$
$$\sigma_{uv} = \Gamma_{12}^1\,\sigma_u+\Gamma_{12}^2\,\sigma_v+\mathbb{M}\,N, \tag{26}$$
$$\sigma_{vv} = \Gamma_{22}^1\,\sigma_u+\Gamma_{22}^2\,\sigma_v+\mathbb{N}\,N, \tag{27}$$

where

$$\begin{aligned}\Gamma_{11}^1&=\frac{\mathbb{G}\,\mathbb{E}_u-2\,\mathbb{F}\,\mathbb{F}_u+\mathbb{F}\,\mathbb{E}_v}{2\,(\mathbb{E}\,\mathbb{G}-\mathbb{F}^2)}, & \Gamma_{11}^2&=\frac{2\,\mathbb{E}\,\mathbb{F}_u-\mathbb{E}\,\mathbb{E}_v+\mathbb{F}\,\mathbb{E}_u}{2\,(\mathbb{E}\,\mathbb{G}-\mathbb{F}^2)},\\ \Gamma_{12}^1&=\frac{\mathbb{G}\,\mathbb{E}_v-\mathbb{F}\,\mathbb{G}_u}{2\,(\mathbb{E}\,\mathbb{G}-\mathbb{F}^2)}, & \Gamma_{12}^2&=\frac{\mathbb{E}\,\mathbb{G}_u-\mathbb{F}\,\mathbb{E}_v}{2\,(\mathbb{E}\,\mathbb{G}-\mathbb{F}^2)},\\ \Gamma_{22}^1&=\frac{2\,\mathbb{G}\,\mathbb{F}_v-\mathbb{G}\,\mathbb{G}_u-\mathbb{F}\,\mathbb{G}_v}{2\,(\mathbb{E}\,\mathbb{G}-\mathbb{F}^2)}, & \Gamma_{22}^2&=\frac{\mathbb{E}\,\mathbb{G}_v-2\,\mathbb{F}\,\mathbb{F}_v+\mathbb{F}\,\mathbb{G}_u}{2\,(\mathbb{E}\,\mathbb{G}-\mathbb{F}^2)}.\end{aligned} \tag{28}$$

The six Γ coefficients in these formulas are called Christoffel symbols.

Remark 17. The formulas (28) look very complicated. However we will see in the proof below that it is not hard to derive them "on the fly".

Proof. First note that as $\{\sigma_u, \sigma_v, N\}$ form a basis of $\mathbb{R}^3$ at p, there must exist nine numbers such that (25–27) hold. Take inner product of (25–27) with N we see that the coefficients for N must be $\mathbb{L}, \mathbb{M}, \mathbb{N}$.

Now consider (25). Taking inner product with σ_u and σ_v we have

$$\mathbb{E}\,\Gamma_{11}^1 + \mathbb{F}\,\Gamma_{11}^2 = \sigma_{uu}\cdot\sigma_u = \left(\frac{\mathbb{E}}{2}\right)_u, \tag{29}$$

$$\mathbb{F}\,\Gamma_{11}^1 + \mathbb{G}\,\Gamma_{11}^2 = \sigma_{uv}\cdot\sigma_v = \left(\frac{\mathbb{G}}{2}\right)_u. \tag{30}$$

The first line of formulas in (28) immediately follows. The proofs for the other four formulas are similar and left as exercise. □

- With the help of Christoffel symbols, we can characterize conditions for a tangent vector field $w(t) := \alpha(t)\,\sigma_u + \beta(t)\,\sigma_v$ to be parallel along a curve $\gamma(t) = \sigma(u(t), v(t))$.

THEOREM 18. [2] *$w(t)$ is parallel along $\gamma(t)$ if and only if the following equations are satisfied:*

$$\begin{aligned} \dot{\alpha} + (\Gamma_{11}^1\,\dot{u} + \Gamma_{12}^1\,\dot{v})\,\alpha + (\Gamma_{12}^1\,\dot{u} + \Gamma_{22}^1\,\dot{v})\,\beta &= 0, \\ \dot{\beta} + (\Gamma_{11}^2\,\dot{u} + \Gamma_{12}^2\,\dot{v})\,\alpha + (\Gamma_{12}^2\,\dot{u} + \Gamma_{22}^2\,\dot{v})\,\beta &= 0. \end{aligned} \tag{31}$$

Proof. This follows easily from (25–27). □

Remark 19. Note that the above equations are easier to remember in matrix form:

$$\dot{\alpha} + \left[\begin{pmatrix} \Gamma_{11}^1 & \Gamma_{12}^1 \\ \Gamma_{12}^1 & \Gamma_{22}^1 \end{pmatrix}\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix}\right]\cdot\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = 0, \tag{32}$$

and

$$\dot{\beta} + \left[\begin{pmatrix} \Gamma_{11}^2 & \Gamma_{12}^2 \\ \Gamma_{12}^2 & \Gamma_{22}^2 \end{pmatrix}\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix}\right]\cdot\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = 0. \tag{33}$$

Remark 20. Also keep in mind that when we "upgrade" to Riemannian geometry, a "surface" will not be given as a "surface patch" with explicit formulas, but as a collection of quantities defined at every $p \in S$: $\mathbb{E}, \mathbb{F}, \mathbb{G}, \mathbb{L}, \mathbb{M}, \mathbb{N}$ and Γ_{ij}^k.

Observe that in (31) only the first fundamental form and the tangent direction $\dot{\gamma}(t)$ are involved.

- Examples and remarks.

Example 21. Let S be the x-y plane, parametrized by $\sigma(u, v) = (u, v, 0)$. Then we easily have $\Gamma_{ij}^k = 0$ for all i, j, k. (31) now becomes

$$\dot{\alpha} = \dot{\beta} = 0. \tag{34}$$

2. Proposition 7.4.5 of the textbook.

Just as we expected.

Remark 22. The Christoffel symbol Γ_{ij}^k is roughly the k-th component of the change of the i-th coordinate vector along the j-th direction.

Note that if $\Gamma_{ij}^k = 0$ for all i, j, k, then the coordinate vectors σ_u, σ_v are parallel along $u = \text{const}$ and $v = \text{const}$.

Example 23. Let S be the cylinder $(\cos u, \sin u, v)$. Then we have

$$\sigma_u = (-\sin u, \cos u, 0), \qquad \sigma_v = (0, 0, 1) \tag{35}$$

and

$$\mathbb{E} = 1, \qquad \mathbb{F} = 0, \qquad \mathbb{G} = 1. \tag{36}$$

Thus $\Gamma_{ij}^k = 0$ for all i, j, k. (31) again gives

$$\dot{\alpha} = \dot{\beta} = 0. \tag{37}$$

Example 24. Let S be the unit sphere $(\cos u \cos v, \cos u \sin v, \sin u)$. We have

$$\mathbb{E} = 1, \qquad \mathbb{F} = 0, \qquad \mathbb{G} = \cos^2 u. \tag{38}$$

This leads to

$$\begin{aligned} \Gamma_{11}^1 &= \frac{\mathbb{G}\,\mathbb{E}_u - 2\,\mathbb{F}\,\mathbb{F}_u + \mathbb{F}\,\mathbb{E}_v}{2\,(\mathbb{E}\,\mathbb{G} - \mathbb{F}^2)} = 0, & \Gamma_{11}^2 &= \frac{2\,\mathbb{E}\,\mathbb{F}_u - \mathbb{E}\,\mathbb{E}_v + \mathbb{F}\,\mathbb{E}_u}{2\,(\mathbb{E}\,\mathbb{G} - \mathbb{F}^2)} = 0, \\ \Gamma_{12}^1 &= \frac{\mathbb{G}\,\mathbb{E}_v - \mathbb{F}\,\mathbb{G}_u}{2\,(\mathbb{E}\,\mathbb{G} - \mathbb{F}^2)} = 0, & \Gamma_{12}^2 &= \frac{\mathbb{E}\,\mathbb{G}_u - \mathbb{F}\,\mathbb{E}_v}{2\,(\mathbb{E}\,\mathbb{G} - \mathbb{F}^2)} = -\tan u, \\ \Gamma_{22}^1 &= \frac{2\,\mathbb{G}\,\mathbb{F}_v - \mathbb{G}\,\mathbb{G}_u - \mathbb{F}\,\mathbb{G}_v}{2\,(\mathbb{E}\,\mathbb{G} - \mathbb{F}^2)} = \sin u \cos u, & \Gamma_{22}^2 &= \frac{\mathbb{E}\,\mathbb{G}_v - 2\,\mathbb{F}\,\mathbb{F}_v + \mathbb{F}\,\mathbb{G}_u}{2\,(\mathbb{E}\,\mathbb{G} - \mathbb{F}^2)} = 0. \end{aligned} \tag{39}$$

(31) now becomes

$$\dot{\alpha} + (\sin u \cos u)\,\dot{v}\,\beta = 0, \qquad \dot{\beta} - (\tan u)\,\dot{v}\,\alpha = 0. \tag{40}$$

Thus w is parallel along γ if and only if (40) holds.

4. Parallel transport map

Definition 25. *Let p, $q \in S$ and let $\gamma(t)$ be a curve on S connecting p, q with $p = \gamma(t_0)$, $q = \gamma(t_1)$. Let $w_0 \in T_p S$. Then there is a unique vector field $w(t)$ parallel along γ with $w(t_0) = w_0$. The map $\Pi_\gamma^{pq}: T_p S \mapsto T_q S$ taking w_0 to $w(t_1)$ is called* parallel transport *from p to q along γ.*

Proposition 26. *Π_γ^{pq} is an isometry.*

Proof. We have

$$\frac{\mathrm{d}}{\mathrm{d}t}(w \cdot \tilde{w}) = \dot{w} \cdot \tilde{w} + w \cdot \dot{\tilde{w}} = 0. \tag{41}$$

as $\dot{w}, \dot{\tilde{w}} \parallel N$. Therefore for $w_0, \tilde{w}_0 \in T_p S$,

$$\Pi_\gamma^{pq}(w_0) \cdot \Pi_\gamma^{pq}(\tilde{w}_0) - w_0 \cdot \tilde{w}_0 = \int_{t_0}^{t_1} \frac{\mathrm{d}}{\mathrm{d}t}(w \cdot \tilde{w}) \, \mathrm{d}t = 0, \tag{42}$$

and the conclusion follows. □

Example 27. Let S be the unit sphere $(\cos u \cos v, \cos u \sin v, \sin u)$. We have seen that a vector field $\alpha(t) \sigma_u + \beta(t) \sigma_v$ is parallel along γ is equivalent to

$$\dot{\alpha} + (\sin u \cos u) \dot{v} \beta = 0, \qquad \dot{\beta} - (\tan u) \dot{v} \alpha = 0. \tag{43}$$

Now notice that unless $\sin u = 0$, that is γ is the big circle, the solution does not satisfy $\dot{\alpha} = \dot{\beta} = 0$.