Math 348 Differential Geometry of Curves and Surfaces

Lecture 14 Minimal and Developable Surfaces

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Midterm Course and Instruction Feedback

- **More examples in class.**
    - Will try;
    - Much of "proofs" etc. can also be read as examples of calculation;
    - A certain percent of homework problems are intentionally designed to be different from examples in class.
- **Use whiteboard instead of slides.**
    - Have made fonts larger – does it help? Please let me know;
    - Slides are posted online before lectures – use phones?
    - Will put less details on slides, just outcome of major steps;
    - Will stop using slides after 2nd midterm if the above do not work.

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Please do not hesitate to interrupt me if you have a question.

Review

Formulas

1. Principal curvatures: $\det \begin{pmatrix} \mathbb{L} - \kappa_i \mathbb{E} & \mathbb{M} - \kappa_i \mathbb{F} \\ \mathbb{M} - \kappa_i \mathbb{F} & \mathbb{N} - \kappa_i \mathbb{G} \end{pmatrix} = 0;$
2. Principal vectors:
$$\left[\begin{pmatrix} \mathbb{L} & \mathbb{M} \\ \mathbb{M} & \mathbb{N} \end{pmatrix} - \kappa_i \begin{pmatrix} \mathbb{E} & \mathbb{F} \\ \mathbb{F} & \mathbb{G} \end{pmatrix} \right] \begin{pmatrix} a_i \\ b_i \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$
$$t_i = a_i \sigma_u + b_i \sigma_v.$$
3. Mean curvature.
$$H = \frac{\kappa_1 + \kappa_2}{2} = \frac{1}{2} Tr \left[\begin{pmatrix} \mathbb{E} & \mathbb{F} \\ \mathbb{F} & \mathbb{G} \end{pmatrix}^{-1} \begin{pmatrix} \mathbb{L} & \mathbb{M} \\ \mathbb{M} & \mathbb{N} \end{pmatrix} \right].$$
4. Gaussian curvature.
$$K = \kappa_1 \kappa_2 = \det \left[\begin{pmatrix} \mathbb{E} & \mathbb{F} \\ \mathbb{F} & \mathbb{G} \end{pmatrix}^{-1} \begin{pmatrix} \mathbb{L} & \mathbb{M} \\ \mathbb{M} & \mathbb{N} \end{pmatrix} \right].$$

Exampes

Calculation of $H, K, \kappa_1, \kappa_2, t_1, t_2$.

$$\boxed{z = \alpha x^2 + \beta y^2,\ H, K, \kappa_1, \kappa_2, t_1, t_2 \text{ at } (0,0,0).}$$

1. Surface patch: $\sigma(u, v) = (u, v, \alpha u^2 + \beta v^2)$;
2. $(0,0,0) = \sigma(0,0)$. So all calculation should be done at $u = v = 0$;
3. Calculate

$$\kappa_1 = 2\alpha, t_1 = (1,0,0); \quad \kappa_2 = 2\beta, t_2 = (0,1,0);$$

$$H = \alpha + \beta; \quad K = 4\alpha\beta.$$

In fact, near every $p \in S$ the surface is approximately $z = \frac{1}{2}(\kappa_1 x^2 + \kappa_2 y^2)$ if we take $T_p S$ to be the $x - y$ plane.

Minimal Surfaces

The problem

$$\boxed{C\text{: Curve. Find surface } S\text{: } \partial S = C\text{, minimal surface area.}}$$

Example

Let C be a simple closed plane curve. Then the minimal surface is the part of the plane enclosed by C.

- Note that we have not been fully rigorous.

Basic idea

$$\boxed{\sigma^0(u,v)\text{: Minimal.}}$$

- $\sigma(u,v) : U \mapsto \mathbb{R}^3$, $\sigma(\partial U) = \{0\}$.
- $\sigma^\tau := \sigma^0 + \tau\sigma$.
- $$A(\tau) := \int_U \|\sigma_u^\tau \times \sigma_v^\tau\| \mathrm{d}u\mathrm{d}v.$$
- We have $A(0) \leqslant A(\tau)$, thus $A'(\tau) = 0$.

Variational calculus

1. $\sigma_u^\tau \times \sigma_v^\tau = V_0 + \tau V_1 + \tau^2 V_2,$
$$V_0 = \sigma_u^0 \times \sigma_v^0, \quad V_1 = \sigma_u^0 \times \sigma_v + \sigma_u \times \sigma_v^0, \quad V_2 = \sigma_u \times \sigma_v.$$
2. $A(\tau) = \int_U \sqrt{V_0 \cdot V_0 + 2\tau V_0 \cdot V_1 + O(\tau^2)}\mathrm{d}u\mathrm{d}v.$
$$\Rightarrow A'(0) = \int_U \frac{V_0 \cdot V_1}{\sqrt{V_0 \cdot V_0}}\mathrm{d}u\mathrm{d}v = \int_U \frac{V_0 \cdot V_1}{\sqrt{\mathbb{E}\mathbb{G} - \mathbb{F}^2}}\mathrm{d}u\mathrm{d}v.$$
3. $V_0 \cdot V_1 = \mathbb{E}(\sigma_v^0 \cdot \sigma_v) - \mathbb{F}(\sigma_u^0 \cdot \sigma_v + \sigma_u \cdot \sigma_v^0) + \mathbb{G}(\sigma_u^0 \cdot \sigma_u).$
4. Simplify.
$$\sigma_u = c_{11}\sigma_u^0 + c_{12}\sigma_v^0 + c_{13}N^0, \qquad \sigma_v = c_{21}\sigma_u^0 + c_{22}\sigma_v^0 + c_{23}N^0.$$
$$V_0 \cdot V_1 = (c_{11} + c_{22})(\mathbb{E}\mathbb{G} - \mathbb{F}^2).$$
5. Let $\sigma(u, v) = f(u, v)N^0(u, v)$. We have
$c_{11} + c_{22} = -2fH.$

Minimal surfaces

$$\boxed{\text{Minimal surface: } H = 0.}$$

Example

1. A plane region is minimal;
2. A spherical region is not minimal;
3. A cylindrical region is not minimal.

Minimal surface of revolution.

$\boxed{\text{If cylinder is not minimal, what is?}}$

Example

1. $\sigma(u,v) = (f(u)\cos v, f(u)\sin v, g(u))$. u: arc length for the plane curve $(f(u), g(u))$. Assume $\dot{g} \neq 0$.
2. $H = \frac{1}{2}\left(\dot{f}\ddot{g} - \ddot{f}\dot{g} + \frac{\dot{g}}{f}\right) = \frac{1}{2}\left(\frac{\dot{g}}{f} - \frac{\ddot{f}}{\dot{g}}\right)$.
3. $H = 0$ becomes $f\ddot{f} = 1 - \dot{f}^2$.
4. Solve the equation: $f = \frac{1}{a}\sqrt{1 + a^2u^2}$.
5. Solve for g.
6. $f = \frac{1}{a}\cosh(a(g - c))$.

Catenoid: Rotating a catenary.

Minimal ruled surfaces.

$$\boxed{\text{The only minimal ruled surfaces are plane and helicoid.}}$$

1. $\sigma(u,v) = \gamma(u) + vl(u)$. Assume $\|l\| = \|\dot{l}\| = 1$.
2. $$H = 0 \Leftrightarrow [\ddot{\gamma} + v\ddot{l} - 2(\dot{\gamma} \cdot l)\dot{l}] \cdot [(\dot{\gamma} + v\dot{l}) \times l] = 0.$$
3. Comparing powers of $1, v, v^2$, obtain three equations.
4. The first is $(\dot{l} \times l) \cdot \ddot{l} = 0$ which leads to $\ddot{l} = -l$.
5. Show that $\dot{\gamma} = 0$.
6. $N = l \times \dot{l}$ constant vector.

   $$\gamma(u) = (f(u), g(u), \gamma_0 u + \gamma_1).$$

   Then solve f, g.

Catenoid $\leftrightarrow$ Helicoid.

Developable Surfaces

Flat surface

$$\boxed{\text{A surface is flat if } K = 0. \text{ Developable} \Leftrightarrow \text{Flat.}}$$

Flat surfaces are ruled.

1. $\sigma(u, v)$ such that

$$\mathbb{E}\mathrm{d}u^2 + \mathbb{G}\mathrm{d}v^2, \qquad \mathbb{L}\mathrm{d}u^2 + \mathbb{N}\mathrm{d}v^2.$$

2. $K = 0 \Rightarrow \mathbb{L} = 0$ or $\mathbb{N} = 0$. Assume $\mathbb{N} = 0$.
3. $N_u = -\mathbb{E}^{-1}\mathbb{L}\sigma_u$, $N_v = 0$.
4. $\gamma(v) := \sigma(u_0, v)$. $\dot{\gamma} \parallel \ddot{\gamma} \Rightarrow$ straight line.

Looking Back and Forward

Summary

$\boxed{\text{Required: §8.1,8.2; Optional: §8.3–8.6.}}$

- **Properties.**
  1. Developable $\Leftrightarrow K = 0$;
  2. Minimal $\Leftrightarrow H = 0$;

See you next Tuesday!

$\boxed{\text{Parallel transport.}}$

1. Motivation and definition;
2. Covariant derivative;
3. Christoffel symbols.