Lectures 14: Curvatures for Surfaces II

Disclaimer. As we have a textbook, this lecture note is for guidance and supplement only. It should not be relied on when preparing for exams.

In this lecture we introduce several quantities that characterize the curving of a surface patch.

The required textbook sections are §8.1–8.2. The optional sections are §8.3–8.6.

I try my best to make the examples in this note different from examples in the textbook. Please read the textbook carefully and try your hands on the exercises. During this please don't hesitate to contact me if you have any questions.

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Principal curvature, mean curvature, Gaussian curvature

- Principal curvatures.

$$\det\begin{pmatrix} \mathbb{L}-\kappa_i\,\mathbb{E} & \mathbb{M}-\kappa_i\,\mathbb{F} \\ \mathbb{M}-\kappa_i\,\mathbb{F} & \mathbb{N}-\kappa_i\,\mathbb{G} \end{pmatrix}=0, \tag{1}$$

$$\left[\begin{pmatrix} \mathbb{L} & \mathbb{M} \\ \mathbb{M} & \mathbb{N} \end{pmatrix}-\kappa_i\begin{pmatrix} \mathbb{E} & \mathbb{F} \\ \mathbb{F} & \mathbb{G} \end{pmatrix}\right]\begin{pmatrix} a_i \\ b_i \end{pmatrix}=0, \tag{2}$$

$$t_i=a_i\,\sigma_u+b_i\,\sigma_v. \tag{3}$$

- Mean curvature.

$$H=\frac{1}{2\,\pi}\int_0^{2\pi}\kappa_n(\theta)\,\mathrm{d}\theta=\frac{\mathbb{E}\,\mathbb{N}+\mathbb{L}\,\mathbb{G}-2\,\mathbb{M}\,\mathbb{F}}{2\,(\mathbb{E}\,\mathbb{G}-\mathbb{F}^2)}=\frac{1}{2}\,\mathrm{Tr}\left[\begin{pmatrix} \mathbb{E} & \mathbb{F} \\ \mathbb{F} & \mathbb{G} \end{pmatrix}^{-1}\begin{pmatrix} \mathbb{L} & \mathbb{M} \\ \mathbb{M} & \mathbb{N} \end{pmatrix}\right]. \tag{4}$$

- Gaussian curvature.

$$K=\lim_{r\longrightarrow 0}\frac{\text{Area of } N(B_r)}{\text{Area of } \sigma(B_r)}=\frac{\mathbb{L}\,\mathbb{N}-\mathbb{M}^2}{\mathbb{E}\,\mathbb{G}-\mathbb{F}^2}=\det\left[\begin{pmatrix} \mathbb{E} & \mathbb{F} \\ \mathbb{F} & \mathbb{G} \end{pmatrix}^{-1}\begin{pmatrix} \mathbb{L} & \mathbb{M} \\ \mathbb{M} & \mathbb{N} \end{pmatrix}\right]. \tag{5}$$

- Relations.

$$H=\frac{\kappa_1+\kappa_2}{2},\qquad K=\kappa_1\,\kappa_2,\qquad \kappa_{1,2}=\frac{H\pm\sqrt{H^2-4\,K}}{2}. \tag{6}$$

$$\kappa_n((\cos\theta)\,t_1+(\sin\theta)\,t_2)=\kappa_1\cos^2\theta+\kappa_2\sin^2\theta. \tag{7}$$

Remark 1. We have seen last time that if $\kappa_1=\kappa_2$ everywhere, then S is part of plane or sphere.

1. Examples

Example 2. Consider the surface $z=\alpha\,x^2+\beta\,y^2$ where $\alpha,\beta\in\mathbb{R}$. Calculate $H,K,\kappa_1,\kappa_2,t_1,t_2$ at the origin.

Solution. We take the surface patch $\sigma(u,v)=(u,v,\alpha\,u^2+\beta\,v^2)$. Then we have

$$\sigma_u=(1,0,2\,\alpha\,u),\qquad \sigma_v=(0,1,2\,\beta\,v),\qquad N=\frac{(-2\,\alpha\,u,-2\,\beta\,v,1)}{\sqrt{1+4\,\alpha^2\,u^2+4\,\beta^2\,v^2}}, \tag{8}$$

$$\sigma_{uu}=(0,0,2\,\alpha),\qquad \sigma_{uv}=(0,0,0),\qquad \sigma_{vv}=(0,0,2\,\beta). \tag{9}$$

Thus at the origin which corresponds to $u=v=0$, we have

$$\mathbb{E}=1,\qquad \mathbb{F}=0,\qquad \mathbb{G}=1, \tag{10}$$

$$\mathbb{L}=2\,\alpha,\qquad \mathbb{M}=0,\qquad \mathbb{N}=2\,\beta. \tag{11}$$

Consequently we have (wlog assume $\alpha>\beta$),

$$\kappa_1=2\,\alpha,\qquad t_1=(1,0,0);\qquad \kappa_2=2\,\beta,\qquad t_2=(0,1,0), \tag{12}$$

$$H=\alpha+\beta,\qquad K=4\,\alpha\,\beta. \tag{13}$$

2. Minimal surfaces (optional)

2.1. The problem

- The so-called "Plateau's problem" asks the following questions: Given a closed curve in the space $\mathbb{R}^3$, among the infinitely many surfaces having this curve as its boundary, which one has the minimal area?

Example 3. Let C be a simple closed plane curve. Then the minimal surface with C as its boundary is the part of the plane enclosed by C.

Proof. Let U be the region of the plane that is enclosed by C. Let $\sigma: U \mapsto \mathbb{R}^3, \sigma(u, v) = (u, v, f(u, v))$ be an arbitrary surface patch. All we need to show is that the area of $\sigma(u, v)$ is no less than the area of U.

Exercise 1. Point out as many gaps as you can in the above set up. Can you fill them?

Now we calculate

$$\sigma_u = (1, 0, f_x), \qquad \sigma_v = (0, 1, f_y) \tag{14}$$

and

$$\sigma_u \times \sigma_v = (-f_x, -f_y, 1). \tag{15}$$

Therefore we have

$$\begin{aligned} \text{Area of } \sigma &= \int_U \|\sigma_u \times \sigma_v\| \, \mathrm{d}u \, \mathrm{d}v \\ &= \int_U \sqrt{1 + f_x^2 + f_y^2} \, \mathrm{d}u \, \mathrm{d}v \\ &= \int_U \mathrm{d}u \, \mathrm{d}v = \text{Area of } U. \end{aligned} \tag{16}$$

Thus ends the proof. □

2.2. Variational analysis

- When the curve is not a plane curve the situation becomes much more complicated.
- We rely one variational analysis to obtain some characterizing equation for this minimal surface.
- Variational analysis is an upgrade of "taking derivative and set it to zero" in first year calculus.
- Let $\sigma^0(u, v): U \mapsto \mathbb{R}^3$ be a surface patch for the minimal surface. Thus we have $\sigma^0(\partial U) = C$. Now let $\sigma(u, v): U \mapsto \mathbb{R}^3$ be an arbitrary surface patch satisfying $\sigma(\partial U) = \{0\}$. Thus at least for $\tau \in \mathbb{R}$ with $|\tau|$ small, we have $\sigma^\tau := \sigma^0 + \tau \sigma$ to be another surface patch with the same boundary C.
- Now define

$$\mathcal{A}(\tau) := \int_U \|\sigma_u^\tau \times \sigma_v^\tau\| \, \mathrm{d}u \, \mathrm{d}v \tag{17}$$

we must have $\mathcal{A}(0) \leqslant \mathcal{A}(\tau)$ for all τ. Consequently we must have $\mathcal{A}'(\tau) = 0$.

- We calculate
$$\sigma_u^\tau = \sigma_u^0 + \tau\,\sigma_u, \qquad \sigma_v^\tau = \sigma_v^0 + \tau\,\sigma_v \tag{18}$$
and therefore
$$\sigma_u^\tau \times \sigma_v^\tau = \sigma_u^0 \times \sigma_v^0 + \tau\,[\sigma_u^0 \times \sigma_v + \sigma_u \times \sigma_v^0] + \tau^2\,\sigma_u \times \sigma_v. \tag{19}$$
Let's denote for now
$$V_0 := \sigma_u^0 \times \sigma_v^0, \qquad V_1 := \sigma_u^0 \times \sigma_v + \sigma_u \times \sigma_v^0, \qquad V_2 := \sigma_u \times \sigma_v. \tag{20}$$

- Thus we have
$$\mathcal{A}(\tau) = \int_U \sqrt{V_0 \cdot V_0 + 2\,\tau\,V_0 \cdot V_1 + O(\tau^2)}\,\mathrm{d}u\,\mathrm{d}v \tag{21}$$
Taking τ-derivative and setting $\tau = 0$ we obtain
$$\mathcal{A}'(0) = \int_U \frac{V_0 \cdot V_1}{\sqrt{V_0 \cdot V_0}}\,\mathrm{d}u\,\mathrm{d}v = \int_U \frac{V_0 \cdot V_1}{\sqrt{\mathbb{E}\,\mathbb{G} - \mathbb{F}^2}}\,\mathrm{d}u\,\mathrm{d}v. \tag{22}$$

- To calculate $V_0 \cdot V_1$ we use the vector identity
$$(a \times b) \cdot (c \times d) = (a \cdot c)\,(b \cdot d) - (a \cdot d)\,(b \cdot c). \tag{23}$$
This leads to
$$\begin{aligned} V_0 \cdot V_1 &= (\sigma_u^0 \times \sigma_v^0) \cdot (\sigma_u^0 \times \sigma_v) + (\sigma_u^0 \times \sigma_v^0) \cdot (\sigma_u \times \sigma_v^0) \\ &= (\sigma_u^0 \cdot \sigma_u^0)\,(\sigma_v^0 \cdot \sigma_v) - (\sigma_u^0 \cdot \sigma_v)\,(\sigma_v^0 \cdot \sigma_u^0) + (\sigma_u^0 \cdot \sigma_u)\,(\sigma_v^0 \cdot \sigma_v^0) - (\sigma_u^0 \cdot \sigma_v^0)\,(\sigma_u \cdot \sigma_v^0) \\ &= \mathbb{E}\,(\sigma_v^0 \cdot \sigma_v) - \mathbb{F}\,(\sigma_u^0 \cdot \sigma_v + \sigma_u \cdot \sigma_v^0) + \mathbb{G}\,(\sigma_u^0 \cdot \sigma_u). \end{aligned} \tag{24}$$

- To simplify (24) we write
$$\sigma_u = c_{11}\,\sigma_u^0 + c_{12}\,\sigma_v^0 + c_{13}\,N^0, \qquad \sigma_v = c_{21}\,\sigma_u^0 + c_{22}\,\sigma_v^0 + c_{23}\,N^0. \tag{25}$$
Therefore
$$\sigma_u \cdot \sigma_u^0 = \mathbb{E}\,c_{11} + \mathbb{F}\,c_{12}, \qquad \sigma_u \cdot \sigma_v^0 = \mathbb{F}\,c_{11} + \mathbb{G}\,c_{12}, \tag{26}$$
$$\sigma_v \cdot \sigma_u^0 = \mathbb{E}\,c_{21} + \mathbb{F}\,c_{22}, \qquad \sigma_v \cdot \sigma_v^0 = \mathbb{F}\,c_{21} + \mathbb{G}\,c_{22}. \tag{27}$$
Substituting into (24) we have
$$V_0 \cdot V_1 = (c_{11} + c_{22})\,(\mathbb{E}\,\mathbb{G} - \mathbb{F}^2). \tag{28}$$
Thus
$$\mathcal{A}'(0) = \int_U (c_{11} + c_{22})\,\sqrt{\mathbb{E}\,\mathbb{G} - \mathbb{F}^2}\,\mathrm{d}u\,\mathrm{d}v \tag{29}$$

- Finally, we notice that as σ is arbitrary, we could restrict ourselves to $\sigma(u, v) = f(u, v)\,N^0(u, v)$ where $f(u, v)$ is a scalar function vanishing on ∂U. Thus we have
$$\sigma_u = f\,N_u^0 + f_u\,N^0, \qquad \sigma_v = f\,N_v^0 + f_v\,N^0. \tag{30}$$

Comparing with (25), and recalling

$$-N_u^0 = a_{11}\,\sigma_u^0 + a_{12}\,\sigma_v^0, \qquad -N_v^0 = a_{21}\,\sigma_u^0 + a_{22}\,\sigma_v^0, \qquad a_{11} + a_{22} = 2\,H, \tag{31}$$

we see that $c_{11} + c_{22} = -2\,f\,H$. Consequently

$$\mathcal{A}'(0) = -2\int_U f\,H\,\sqrt{\mathbb{E}\,\mathbb{G} - \mathbb{F}^2}\,\mathrm{d}u\,\mathrm{d}v = -2\int_{S^0} f\,H\,\mathrm{d}S \tag{32}$$

where the last is the surface integral as defined in multivariable calculus.

- Since f is arbitrary, for $\mathcal{A}'(0) = 0$ we must have $H = 0$.

Definition 4. (Minimal surface) [1]*A* ***minimal surface*** *is a surface whose mean curvature is zero everywhere.*

2.3. Examples

Example 5. A plane region is a minimal surface; A spherical region is not minimal; A cylindrical region is also not minimal.

Proof.

- Plane. $\sigma(u,v) = (u,v)$. As $\sigma_{uu} = \sigma_{uv} = \sigma_{vv} = 0$, we have $\mathbb{L} = \mathbb{M} = \mathbb{N} = 0$. Consequently $H = \frac{1}{2}\mathrm{Tr}\left[\begin{pmatrix} \mathbb{E} & \mathbb{F} \\ \mathbb{F} & \mathbb{G} \end{pmatrix}^{-1}\begin{pmatrix} \mathbb{L} & \mathbb{M} \\ \mathbb{M} & \mathbb{N} \end{pmatrix}\right] = 0$.
- Sperical. Let S be part of a sphere of radius R centered at the origin. Then we easily see that

$$N_u = R^{-1}\,\sigma_u, \qquad N_v = R^{-1}\,\sigma_v. \tag{33}$$

Thus we have $\mathbb{L} = -R^{-1}\,\mathbb{E}, \mathbb{M} = -R^{-1}\,\mathbb{F}, \mathbb{N} = -R^{-1}\,\mathbb{G}$, and

$$H = \frac{1}{2}\mathrm{Tr}\left[\begin{pmatrix} \mathbb{E} & \mathbb{F} \\ \mathbb{F} & \mathbb{G} \end{pmatrix}^{-1}\begin{pmatrix} \mathbb{L} & \mathbb{M} \\ \mathbb{M} & \mathbb{N} \end{pmatrix}\right] = \frac{1}{2}\,\mathrm{Tr}\begin{pmatrix} -R^{-1} & 0 \\ 0 & -R^{-1} \end{pmatrix} = -\frac{1}{R}. \tag{34}$$

- Cylinder. $\sigma(u,v) = (\cos u, \sin u, v)$. We have

$$\sigma_u = (-\sin u, \cos u, 0), \quad \sigma_v = (0,0,1), \tag{35}$$

$$N = (\cos u, \sin u, 0), \qquad N_u = (-\sin u, \cos u, 0), \quad N_v = (0,0,0). \tag{36}$$

Thus

$$\mathbb{E} = 1, \mathbb{F} = 0, \mathbb{G} = 1; \qquad \mathbb{L} = 1, \mathbb{M} = 0, \mathbb{N} = 0. \tag{37}$$

Consequently

$$H = \frac{1}{2}\,\mathrm{Tr}\left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}^{-1}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right] = \frac{1}{2}. \tag{38}$$

□

1. Definition 12.1.2 in the textbook.

Example 6. The only minimal surfaces of revolution are plane and catenoid.

Proof. Let $\sigma(u,v)=(f(u)\cos v, f(u)\sin v, g(u))$ be the surface of revolution generated by the plane curve $(f(u), g(u))$. We assume that u is the arc length parameter of this plane curve, that is

$$\dot{f}^2+\dot{g}^2=1. \tag{39}$$

We further assume that $\dot{g}\neq 0$. Note that this excludes the "plane" case. We also assume that $f>0$.

1. Calculate H.

$$\sigma_u=\left(\dot{f}(u)\cos v, \dot{f}(u)\sin v, \dot{g}(u)\right), \qquad \sigma_v=(-f(u)\sin v, f(u)\cos v, 0). \tag{40}$$

$$N=\frac{\sigma_u\times\sigma_v}{\|\sigma_u\times\sigma_v\|}=\left(-\dot{g}(u)\cos v, -\dot{g}(u)\sin v, \dot{f}(u)\right). \tag{41}$$

$$\sigma_{uu}=\left(\ddot{f}(u)\cos v, \ddot{f}(u)\sin v, \ddot{g}(u)\right), \sigma_{uv}=\left(-\dot{f}(u)\sin v, \dot{f}(u)\cos v, 0\right), \tag{42}$$

$$\sigma_{vv}=(-f(u)\cos v, -f(u)\sin v, 0). \tag{43}$$

Thus

$$\mathbb{E}=1, \quad \mathbb{F}=0, \quad \mathbb{G}=f^2(u), \tag{44}$$

$$\mathbb{L}=\dot{f}(u)\,\ddot{g}(u)-\ddot{f}(u)\,\dot{g}(u), \quad \mathbb{M}=0, \quad \mathbb{N}=f(u)\,\dot{g}(u). \tag{45}$$

So

$$H=\frac{1}{2}\operatorname{Tr}\left[\begin{pmatrix}1 & 0\\ 0 & f^2\end{pmatrix}^{-1}\begin{pmatrix}\dot{f}\ddot{g}-\ddot{f}\dot{g} & 0\\ 0 & f\dot{g}\end{pmatrix}\right]=\frac{1}{2}\left(\dot{f}\ddot{g}-\ddot{f}\dot{g}+\frac{\dot{g}}{f}\right). \tag{46}$$

2. Equation for f. Setting $H=0$ we have

$$f\dot{f}\ddot{g}-f\ddot{f}\dot{g}+\dot{g}=0. \tag{47}$$

Now differentiating $\dot{f}^2+\dot{g}^2=1$ we obtain $\dot{f}\ddot{f}+\dot{g}\ddot{g}=0\Longrightarrow\ddot{g}=-\frac{\dot{f}\ddot{f}}{\dot{g}}$. Substituting this into (47) we obtain

$$f\ddot{f}=\dot{g}^2=1-\dot{f}^2. \tag{48}$$

3. Solve (48). We have

$$1=f\ddot{f}+\dot{f}^2=\frac{\mathrm{d}}{\mathrm{d}u}(f\dot{f})=\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}u^2}(f^2). \tag{49}$$

Therefore

$$f^2=u^2+c_1u+c_2. \tag{50}$$

Some change of variables gives

$$f(u)=\frac{1}{a}\sqrt{1+a^2u^2} \tag{51}$$

for some parameter a.

4. Solve for g. We have

$$\dot{g}^2 = 1 - \dot{f}^2 = \frac{1}{1 + a^2 u^2} \Longrightarrow \dot{g} = \pm \frac{1}{\sqrt{1 + a^2 u^2}}. \tag{52}$$

To solve the equation we set $G(v) = g(u)$ where $v = a\,u$. Thus

$$\dot{G} = \frac{1}{a}\,\dot{g}(v/a) = \pm \frac{1}{a}\,\frac{1}{\sqrt{1 + v^2}}. \tag{53}$$

Solving this we have

$$G(v) = \pm \frac{1}{a} \sinh^{-1} v \Longrightarrow g(u) - c = \pm \frac{1}{a} \sinh^{-1}(a\,u). \tag{54}$$

5. Finally we have

$$f = \frac{1}{a}\sqrt{1 + (a\,u)^2} = \frac{1}{a}\sqrt{1 + \sinh^2(a\,(g - c))} = \frac{1}{a}\cosh(a\,(g - c)). \tag{55}$$

This is the equation of a catenary and the surface is then a catenoid. □

Exercise 2. Solve the case $\dot{g} = 0$.

Example 7. [2]Any ruled minimal surface is an open subset of a plane or a helicoid.

Proof. Let $\sigma(u, v) = \gamma(u) + v\, l(u)$ be a surface patch for the ruled minimal surface. We calculate

$$\begin{array}{lll} \sigma_u = \dot{\gamma} + v\,\dot{l}, & \sigma_v = l, & \sigma_u \times \sigma_v = (\dot{\gamma} + v\,\dot{l}) \times l \\ \sigma_{uu} = \ddot{\gamma} + v\,\ddot{l}, & \sigma_{uv} = \dot{l}, & \sigma_{vv} = 0 \end{array}. \tag{56}$$

Therefore

$$\begin{array}{lll} \mathbb{E} = (\dot{\gamma} + v\,\dot{l}) \cdot (\dot{\gamma} + v\,\dot{l}), & \mathbb{F} = (\dot{\gamma} + v\,\dot{l}) \cdot l, & \mathbb{G} = l \cdot l, \\ \mathbb{L} = \dfrac{(\ddot{\gamma} + v\,\ddot{l}) \cdot [(\dot{\gamma} + v\,\dot{l}) \times l]}{\|(\dot{\gamma} + v\,\dot{l}) \times l\|}, & \mathbb{M} = \dfrac{\dot{l} \cdot [(\dot{\gamma} + v\,\dot{l}) \times l]}{\|(\dot{\gamma} + v\,\dot{l}) \times l\|}, & \mathbb{N} = 0. \end{array} \tag{57}$$

Now we make simplifying assumptions.

- It is clear that we can assume $\|l(u)\| = 1$. This simplifies (57) to

$$\begin{array}{lll} \mathbb{E} = (\dot{\gamma} + v\,\dot{l}) \cdot (\dot{\gamma} + v\,\dot{l}), & \mathbb{F} = \dot{\gamma} \cdot l, & \mathbb{G} = 1, \\ \mathbb{L} = \dfrac{(\ddot{\gamma} + v\,\ddot{l}) \cdot [(\dot{\gamma} + v\,\dot{l}) \times l]}{\|(\dot{\gamma} + v\,\dot{l}) \times l\|}, & \mathbb{M} = \dfrac{\dot{l} \cdot [(\dot{\gamma} + v\,\dot{l}) \times l]}{\|(\dot{\gamma} + v\,\dot{l}) \times l\|}, & \mathbb{N} = 0. \end{array} \tag{58}$$

- We can further assume $\|\dot{l}(u)\| = 1$.

Now $H = 0$ implies $\mathbb{L}\,\mathbb{G} - 2\,\mathbb{M}\,\mathbb{F} = 0$ which becomes

$$[\ddot{\gamma} + v\,\ddot{l} - 2\,(\dot{\gamma} \cdot l)\,\dot{l}] \cdot [(\dot{\gamma} + v\,\dot{l}) \times l] = 0. \tag{59}$$

2. Proposition 12.2.4 in the textbook.

Expanding (59) into powers of v, we see that

$$\left[\left(\dot{l}\times l\right)\cdot\ddot{l}\,\right]v^2+\left[\left(\dot{l}\times l\right)\cdot\ddot{\gamma}+(\dot{\gamma}\times l)\cdot\ddot{l}\,\right]v+\left[(\dot{\gamma}\times l)\cdot\ddot{\gamma}-2\,(\dot{\gamma}\cdot l)\left((\dot{\gamma}\times l)\cdot\dot{l}\right)\right]=0. \tag{60}$$

(60) must hold for all v. Consequently

$$\begin{aligned}\left(\dot{l}\times l\right)\cdot\ddot{l} &= 0, &(61)\\ \left(\dot{l}\times l\right)\cdot\ddot{\gamma}+(\dot{\gamma}\times l)\cdot\ddot{l} &= 0, &(62)\\ (\dot{\gamma}\times l)\cdot\ddot{\gamma}-2\,(\dot{\gamma}\cdot l)\left((\dot{\gamma}\times l)\cdot\dot{l}\right) &= 0. &(63)\end{aligned}$$

Now by (61) we conclude that $\ddot{l}(u)$ is a linear combination of $\dot{l}$ and l. Now

$$\|l\|=1\Longrightarrow l\perp\dot{l};\qquad \|\dot{l}\|=1\Longrightarrow\ddot{l}\perp\dot{l}, \tag{64}$$

therefore $\ddot{l}\parallel l$. In fact, differentiating $l\cdot\dot{l}=0$ we have $\ddot{l}\cdot l=-\|\dot{l}\|^2=-1$. Therefore $\ddot{l}=-l$.

Now notice that $\{l,\ \dot{l},\ N=l\times\dot{l}\}$ form an orthonormal basis. Thus we write $\dot{\gamma}=\lambda\,l+\mu\,\dot{l}+\gamma\,N$. Taking derivative and using the facts that N is a constant vector[3] as well as $\ddot{l}=-l$, we have

$$\ddot{\gamma}=\left(\dot{\lambda}-\mu\right)l+(\lambda+\dot{\mu})\,\dot{l}+\dot{\gamma}\,N. \tag{65}$$

By (62) we have $\left(\dot{l}\times l\right)\cdot\ddot{\gamma}=0$ which means $\dot{\gamma}=0$ so $\gamma=\gamma_0$ is a constant.

Finally we take $\operatorname{span}\{l,\dot{l}\}$ to be the x-y plane. Thus we have

$$\gamma(u)=(f(u),g(u),\gamma_0\,u+\gamma_1) \tag{66}$$

where γ_0,γ_1 are constants, and $l(u)=(\cos u,\sin u,0)$. Now there are two cases.

- $\gamma_0=0$. Clearly σ is part of a plane (recall that l also is in the x-y plane);
- $\gamma_0\neq 0$. In this case (63) simplifies to

$$\ddot{g}\cos u-\ddot{f}\sin u=2\left(\dot{f}\cos u+\dot{g}\sin u\right). \tag{67}$$

Now notice that we can always pick $\gamma(u)$ such that $\dot{\gamma}(u)\cdot l(u)=0$. This gives $\dot{f}\cos u+\dot{g}\sin u=0$ and consequently

$$\frac{\mathrm{d}}{\mathrm{d}u}\left(\dot{g}\cos u-\dot{f}\sin u\right)=0\Longrightarrow\dot{g}\cos u-\dot{f}\sin u=c_0 \tag{68}$$

Putting together

$$\begin{aligned}\dot{f}\cos u+\dot{g}\sin u &= 0 &(69)\\ -\dot{f}\sin u+\dot{g}\cos u &= c_0 &(70)\end{aligned}$$

we reach

$$\dot{f}=-c_0\sin u,\qquad \dot{g}=c_0\cos u \tag{71}$$

which means

$$f=c_1+c_0\cos u,\qquad g=c_2+c_0\sin u. \tag{72}$$

3. $\dot{N}=\dot{l}\times\dot{l}+l\times\ddot{l}=0$.

So finally we have

$$\sigma(u,v) = (c_1 + (v + c_0)\cos u, c_2 + (v + c_0)\sin u, \gamma_0 u + \gamma_1) \tag{73}$$

which is the same as

$$\sigma(u,v) = (c_1 + v\,\mathrm{cosu}, c_2 + v\,\mathrm{sinu}, \gamma_1 + \gamma_0\, u), \tag{74}$$

a helicoid. □

Remark 8. Note that the helicoid is not developable.

3. Developable surfaces (optional)

Recall that we have proved that the only developable surfaces are the plane, the (generalized) cylinder, the (generalized) cone, and a class of surfaces called "tangent developables". In the proof we left one big gap: the claim that any developable surface must be ruled. Now we finally are able to fill (half of) this gap.

In the following we assume S is a developable surface, that is a surface having local isometries with the flat plane. Recall that a local isometry $f: S_1 \mapsto S_2$ is characterized by the fact that for every surface patch σ_1 for S_1, if we denote by $\sigma_2 := f \circ \sigma_1$, then the first fundamental forms are identical: $\mathbb{E}_1 = \mathbb{E}_2, \mathbb{F}_1 = \mathbb{F}_2, \mathbb{G}_1 = \mathbb{G}_2$.

LEMMA 9. *S must have Gaussian curvature zero everywhere.*

Proof. Will be proved in a later lecture. □

DEFINITION 10. *A surface S is said to be flat if its Gaussian curvature is zero everywhere.*

PROPOSITION 11. (PROPOSITION 8.4.2 OF THE TEXTBOOK) *Let S be a flat surface with the principal curvatures $\kappa_1 \neq \kappa_2$ everywhere. Then S is a ruled surface.*

Proof.

i. Pick $\sigma(u,v)$ such that the first and second fundamental forms are

$$\mathbb{E}\,\mathrm{d}u^2 + \mathbb{G}\,\mathrm{d}v^2, \qquad \mathbb{L}\,\mathrm{d}u^2 + \mathbb{N}\,\mathrm{d}v^2. \tag{75}$$

Exercise 3. Why can this be done?

ii. Since $K = 0$, there must hold $\mathbb{L}\,\mathbb{N} = 0$. Note that if both $\mathbb{L}, \mathbb{N} = 0$, then $\kappa_1 = \kappa_2 = 0$. Therefore we can assume $\mathbb{L} \neq 0$ or $\mathbb{N} \neq 0$. We study the case $\mathbb{L} \neq 0$ and leave the case $\mathbb{N} \neq 0$ as exercise. Note that if $\mathbb{L} \neq 0$ then necessarily $\mathbb{N} = 0$.

The second fundamental form is now $\mathbb{L}\,\mathrm{d}u^2$.

iii. We have

$$\begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix} = \begin{pmatrix} \mathbb{E} & 0 \\ 0 & \mathbb{G} \end{pmatrix}^{-1} \begin{pmatrix} \mathbb{L} & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \frac{\mathbb{L}}{\mathbb{E}} & 0 \\ 0 & 0 \end{pmatrix} \tag{76}$$

which gives

$$N_u = -\mathbb{E}^{-1}\,\mathbb{L}\,\sigma_u, \qquad N_v = 0. \tag{77}$$

Note that this implies $N_{uv} = N_{vu} = 0$.

iv. Let $p_0 = \sigma(u_0, v_0)$ be arbitrary. We will prove that $\gamma(v) := \sigma(u_0, v)$ is a straight line. We have

$$\dot{\gamma} = \sigma_v, \qquad \ddot{\gamma} = \sigma_{vv}. \tag{78}$$

Now as $\mathbb{N} = 0$ we have $\sigma_{vv} \cdot N = 0$. On the other hand,

$$\sigma_{vv} \cdot \sigma_u = \mathbb{F}_v - \sigma_{uv} \cdot \sigma_v = (\mathbb{L}^{-1}\,\mathbb{E}\,N_u)_v \cdot \sigma_v = \mathbb{L}^{-1}\,\mathbb{E}\,N_{uv} \cdot \sigma_v = 0. \tag{79}$$

Thus we see that $\dot{\gamma} \parallel \ddot{\gamma}$ which means γ is a straight line. $\square$