Math 348 Differential Geometry of Curves and Surfaces

Lecture 13 Curvatures

Xinwei Yu

Oct. 24, 2017

CAB 527, xinwei2@ualberta.ca
Department of Mathematical & Statistical Sciences
University of Alberta

Table of contents

Please do not hesitate to interrupt me if you have a question.

Review

The second fundamental form

- **The second fundamental form.**

$$\langle\!\langle v, w \rangle\!\rangle_{p,S} = \mathbb{L} v_1 w_1 + \mathbb{M}(v_1 w_2 + v_2 w_1) + \mathbb{N} v_2 w_2.$$

$\mathbb{L} = \sigma_{uu} \cdot N = -N_u \cdot \sigma_u$, $\mathbb{M} = \sigma_{uv} \cdot N = -N_u \cdot \sigma_v = -N_v \cdot \sigma_u$,
$\mathbb{N} = \sigma_{vv} \cdot N = -N_v \cdot \sigma_v$.

- **The Weingarten map.** $\mathcal{W}_{p,S} := -D_p \mathcal{G}$

$$\mathcal{W}_{p,S}(a\sigma_u + b\sigma_v) = \tilde{a}\sigma_u + \tilde{b}\sigma_v, \quad \begin{pmatrix} \tilde{a} \\ \tilde{b} \end{pmatrix} = \begin{pmatrix} \mathbb{E} & \mathbb{F} \\ \mathbb{F} & \mathbb{G} \end{pmatrix}^{-1} \begin{pmatrix} \mathbb{L} & \mathbb{M} \\ \mathbb{M} & \mathbb{N} \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}$$

- **Relation between the first and second fundamental forms.**

$$\langle\!\langle v, w \rangle\!\rangle_{p,S} = \langle \mathcal{W}_{p,S}(v), w \rangle_{p,S} = \langle v, \mathcal{W}_{p,S}(w) \rangle_{p,S}.$$

- **Normal curvature and geodesic curvature.**
  - When $\|w\| = 1$, $\kappa_n(p, w) = \mathbb{L} w_1^2 + 2\mathbb{M} w_1 w_2 + \mathbb{N} w_2^2$.
  - $\kappa N = \kappa_n N_S + \kappa_g (N_S \times T)$, $\kappa^2 = \kappa_n^2 + \kappa_g^2$.
- **Geodesic equations.**

Curvatures

Mean curvature.

$p \in S$. $w_0 \in T_pS$, unit vector.

- $w \in T_pS$, unit vector;
- Normal curvature $\kappa_n(p, w)$ can be seen as a function of the angle θ from w_0 to w. $\kappa_n = \kappa_n(\theta)$.
- Taking average $\Rightarrow$ "mean" curvature:

$$H := \frac{1}{2\pi} \int_0^{2\pi} \kappa_n(\theta) \mathrm{d}\theta.$$

- H is independent of the choice of w_0.
- H is a property of S at the point p.

Gaussian curvature.

$p \in S$. $\mathcal{G}$: Gauss map.

- Let σ be a surface patch for S;
- $N = \mathcal{G} \circ \sigma$ is a surface patch for $\mathbb{S}^2$;
- Let $\Omega = \sigma(U)$ be a region in S. Then

$$\text{Area of } \Omega = \int_U \|\sigma_u \times \sigma_v\| \mathrm{d}u \mathrm{d}v,$$

$$\text{Area of } \mathcal{G}(\Omega) = \int_U \|N_u \times N_v\| \mathrm{d}u \mathrm{d}v.$$

- Define the Gaussian curvature $K = \frac{\mathbb{LN}-\mathbb{M}^2}{\mathbb{EG}-\mathbb{F}^2}$. Then

$$\|N_u \times N_v\| = |K| \|\sigma_u \times \sigma_v\|.$$

- $$|K(p)| = \lim_{U \to \{p\}} \frac{\text{Area of } \mathcal{G}(\Omega)}{\text{Are of } \Omega}.$$

Principal curvatures and principal vectors.

$p \in S$. $\kappa_n(\theta)$: normal curvatures of p.

- Let e_1, e_2 be an orthonormal basis for T_pS;
- Can assume $\theta =$ angle from e_1;
- Formula for normal curvatures.

$$\begin{aligned} \kappa_n(\theta) &= \kappa_n(\cos\theta e_1 + \sin\theta e_2) \\ &= \langle\langle e_1, e_1\rangle\rangle \cos^2\theta + 2\langle\langle e_1, e_2\rangle\rangle \cos\theta\sin\theta + \langle\langle e_2, e_2\rangle\rangle \sin^2\theta. \end{aligned}$$

- Four critical points: $\theta_0, \theta_0 + \pi/2, \theta_0 + \pi, \theta_0 + 3\pi/2$.
- Two (equal) maxima κ_1 and two (equal) minima κ_2. Call them the "principal curvatures".
- Two "principal directions": $\kappa(\pm t_1) = \kappa_1, \kappa(\pm t_2) = \kappa_2$.
- t_1, t_2 are orthogonal. Thus $H = \frac{\kappa_1+\kappa_2}{2}$.

Calculation of curvatures.

- **Gaussian curvature.** $K = \frac{\mathbb{LN}-\mathbb{M}^2}{\mathbb{EG}-\mathbb{F}^2}$.
- **Mean curvature and principal curvatures.**
  1. Alternative calculation of the principal curvatures.

     $$\max(\textit{or}\ \min)\kappa(w) = \mathbb{L}w_1^2+2\mathbb{M}w_1w_2+\mathbb{N}w_2^2 \text{ s. t. } \mathbb{E}w_1^2+2\mathbb{F}w_1w_2+\mathbb{G}w_2^2 = 1.$$

  2. Apply the theory of Lagrange multipliers.

     $$L(a,b) = \left[\mathbb{L}w_1^2 + 2\mathbb{M}w_1w_2 + \mathbb{N}w_2^2\right] - \lambda\left[\mathbb{E}w_1^2 + 2\mathbb{F}w_1w_2 + \mathbb{G}w_2^2 - 1\right].$$

     We have

     $$\left[\begin{pmatrix}\mathbb{L} & \mathbb{M}\\ \mathbb{M} & \mathbb{N}\end{pmatrix} - \lambda\begin{pmatrix}\mathbb{E} & \mathbb{F}\\ \mathbb{F} & \mathbb{G}\end{pmatrix}\right]\begin{pmatrix}w_1\\ w_2\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix}.$$

  3. $\lambda = \frac{\langle\langle w,w\rangle\rangle}{\langle w,w\rangle} = \kappa_1, \kappa_2$, corresponding $w_1\sigma_u + w_2\sigma_v$ are t_1, t_2.

Calculation of $H, K, \kappa_1, \kappa_2, t_1, t_2$.

Fundamental forms: $\mathbb{E}\mathrm{d}u^2 + 2\mathbb{F}\mathrm{d}u\mathrm{d}v + \mathbb{G}\mathrm{d}v^2$, $\mathbb{L}\mathrm{d}u^2 + 2\mathbb{M}\mathrm{d}u\mathrm{d}v + \mathbb{N}\mathrm{d}v^2$.

1. Mean curvature and Gaussian curvature.

$$H = \frac{1}{2} Tr \left[\begin{pmatrix} \mathbb{E} & \mathbb{F} \\ \mathbb{F} & \mathbb{G} \end{pmatrix}^{-1} \begin{pmatrix} \mathbb{L} & \mathbb{M} \\ \mathbb{M} & \mathbb{N} \end{pmatrix} \right], \ K = \det \left[\begin{pmatrix} \mathbb{E} & \mathbb{F} \\ \mathbb{F} & \mathbb{G} \end{pmatrix}^{-1} \begin{pmatrix} \mathbb{L} & \mathbb{M} \\ \mathbb{M} & \mathbb{N} \end{pmatrix} \right]$$

2. Principal curvatures and principal vectors. Solve

$$\left[\begin{pmatrix} \mathbb{L} & \mathbb{M} \\ \mathbb{M} & \mathbb{N} \end{pmatrix} - \lambda \begin{pmatrix} \mathbb{E} & \mathbb{F} \\ \mathbb{F} & \mathbb{G} \end{pmatrix} \right] \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

   Solutions are κ_1, a_1, b_1, κ_2, a_2, b_2. Principal vectors are $t_i = a_i \sigma_u + b_i \sigma_v$.

3. $H = \frac{\kappa_1 + \kappa_2}{2}, K = \kappa_1 \kappa_2$.

Examples

Graph

$$\boxed{\sigma(u,v) = (u, v, f(u,v)).}$$

$$K = \frac{f_{xx}f_{yy} - f_{xy}^2}{(1 + f_x^2 + f_y^2)^2}, H = \frac{(1 + f_y^2)f_{xx} - 2f_x f_y f_{xy} + (1 + f_x^2)f_{yy}}{2(1 + f_x^2 + f_y^2)^{3/2}}.$$

Looking Back and Forward

Summary

Required: §8.1,8.2; Optional: §8.3–8.6.

- **Properties.**
  1. Gaussian curvature: Limiting ratio of areas;
  2. Mean curvature: angular average of normal curvatures;
  3. Principal curvatures: maximum and minimum of normal curvatures.
  4. Principal vectors: Directions along which the normal curvature equals principal curvatures.
- **Formulas and procedures.**
  1. Principal curvatures: $\det \begin{pmatrix} \mathbb{L} - \kappa_i \mathbb{E} & \mathbb{M} - \kappa_i \mathbb{F} \\ \mathbb{M} - \kappa_i \mathbb{F} & \mathbb{N} - \kappa_i \mathbb{G} \end{pmatrix} = 0$;
  2. Principal vectors: $\left[\begin{pmatrix} \mathbb{L} & \mathbb{M} \\ \mathbb{M} & \mathbb{N} \end{pmatrix} - \kappa_i \begin{pmatrix} \mathbb{E} & \mathbb{F} \\ \mathbb{F} & \mathbb{G} \end{pmatrix} \right] \begin{pmatrix} a_i \\ b_i \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.,
     $t_i = a_i \sigma_u + b_i \sigma_v$.
  3. Mean curvature. $H = \frac{\kappa_1 + \kappa_2}{2} = \frac{1}{2} Tr \left[\begin{pmatrix} \mathbb{E} & \mathbb{F} \\ \mathbb{F} & \mathbb{G} \end{pmatrix}^{-1} \begin{pmatrix} \mathbb{L} & \mathbb{M} \\ \mathbb{M} & \mathbb{N} \end{pmatrix} \right]$.
  4. Gaussian curvature. $K = \kappa_1 \kappa_2 = \det \left[\begin{pmatrix} \mathbb{E} & \mathbb{F} \\ \mathbb{F} & \mathbb{G} \end{pmatrix}^{-1} \begin{pmatrix} \mathbb{L} & \mathbb{M} \\ \mathbb{M} & \mathbb{N} \end{pmatrix} \right]$.

See you this Thursday!

Understanding surfaces through curvatures.

1. More examples.
2. Minimal surfaces.
3. Developable surfaces;