Lectures 13: Curvatures for Surfaces I

Disclaimer. As we have a textbook, this lecture note is for guidance and supplement only. It should not be relied on when preparing for exams.

In this lecture we introduce several quantities that characterize the curving of a surface patch.

The required textbook sections are §8.1–8.2. The optional sections are §8.3–8.6.

I try my best to make the examples in this note different from examples in the textbook. Please read the textbook carefully and try your hands on the exercises. During this please don't hesitate to contact me if you have any questions.

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1. Gaussian and mean curvatures

- One can show that the first and second fundamental forms completely determines the surface.
- However these are complicated quantities. It turns out that there are more compact ways to understand the curving of surfaces.
- Mean curvature. Consider the normal curvatures κ_n at one point. Pick an arbitrary direction $w_0 \in T_p S$ and let θ be the counterclockwise angle from w_0 to the tangent direction w along which κ_n is calculated. Then we have $\kappa_n = \kappa_n(\theta)$. We will define the mean curvature as the average of all the κ_n's:

$$H := \frac{1}{2\pi} \int_0^{2\pi} \kappa_n(\theta)\, \mathrm{d}\theta. \tag{1}$$

Remark 1. It is important to realize that H is independent of the choice of w_0. That is, if we take another $w_1 \in T_p S$ and let θ_1 be the angle from w_1 to w, we have

$$\frac{1}{2\pi} \int_0^{2\pi} \kappa_n(\theta_1)\, \mathrm{d}\theta_1 = \frac{1}{2\pi} \int_0^{2\pi} \kappa_n(\theta)\, \mathrm{d}\theta = H. \tag{2}$$

Exercise 1. Prove this.

- Gaussian curvature. Consider the Gauss map $\mathcal{G}: S \mapsto \mathbb{S}^2$ and the corresponding Weingarten map $\mathcal{W}$. Recall that

$$\mathcal{W}(\sigma_u) = -N_u = a_{11}\sigma_u + a_{12}\sigma_v, \qquad \mathcal{W}(\sigma_v) = -N_v = a_{21}\sigma_u + a_{22}\sigma_v, \tag{3}$$

where $a_{11}, \ldots, a_{22}$ can be calculated through

$$\begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix} = \begin{pmatrix} \mathbb{E} & \mathbb{F} \\ \mathbb{F} & \mathbb{G} \end{pmatrix}^{-1} \begin{pmatrix} \mathbb{L} & \mathbb{M} \\ \mathbb{M} & \mathbb{N} \end{pmatrix}. \tag{4}$$

Now let U be a region in the u-v plane. Then $N: U \mapsto \mathbb{S}^2$ is a surface patch for $\mathbb{S}^2$. We calculate

$$N_u \times N_v = (a_{11}a_{22} - a_{21}a_{12})\, \sigma_u \times \sigma_v. \tag{5}$$

Therefore

$$\|N_u \times N_v\| = |a_{11}a_{22} - a_{21}a_{12}|\, \|\sigma_u \times \sigma_v\|. \tag{6}$$

Consequently

$$\text{Area of } N(U) = \int_U |a_{11}a_{22} - a_{21}a_{12}|\, \|\sigma_u \times \sigma_v\|\, \mathrm{d}u\, \mathrm{d}v \tag{7}$$

and if we take U_r to be a small disc $D_p((u_0, v_0))$ centering at (u_0, v_0) with radius r, we would have

$$\lim_{r \longrightarrow 0} \frac{\text{Area of } N(U)}{\text{Area of } \sigma(U)} = |a_{11}a_{22} - a_{21}a_{12}|. \tag{8}$$

Exercise 2. Prove this.

We will call the number

$$K := a_{11}\,a_{22} - a_{21}\,a_{12} \tag{9}$$

the Gaussian curvature of S at p_0.

2. Principal curvatures

- We try to understand the mean curvature H. To do this we need a formula for $\kappa_n(\theta)$.
- Recall that if we take $\|w(\theta)\| = 1$,

$$\kappa_n(\theta) = \frac{\langle\langle w(\theta), w(\theta)\rangle\rangle}{\langle w(\theta), w(\theta)\rangle} = \langle\langle w(\theta), w(\theta)\rangle\rangle. \tag{10}$$

- Now let e_1, e_2 be an orthonormal basis for the tangent plane T_pS, we can set $w(\theta) = \cos\theta\, e_1 + \sin\theta\, e_2$. Substituting into (10) we have

$$\kappa_n(\theta) = \langle\langle e_1, e_1\rangle\rangle \cos^2\theta + 2\,\langle\langle e_1, e_2\rangle\rangle \cos\theta \sin\theta + \langle\langle e_2, e_2\rangle\rangle \sin^2\theta. \tag{11}$$

  Integrating we get

$$H = \frac{1}{2\,\pi}\int_0^{2\pi} \kappa_n(\theta)\,\mathrm{d}\theta = \frac{1}{2}\,[\langle\langle e_1, e_1\rangle\rangle + \langle\langle e_2, e_2\rangle\rangle\,]. \tag{12}$$

- Taking derivative

$$\kappa_n'(\theta) = (\langle\langle e_2, e_2\rangle\rangle - \langle\langle e_1, e_1\rangle\rangle)\cos 2\,\theta + 2\,\langle\langle e_1, e_2\rangle\rangle \sin 2\,\theta. \tag{13}$$

  We see that $\kappa_n'(\theta) = 0$ has four solutions in $[0, 2\,\pi]$: $\theta_0, \theta_0 + \pi/2, \theta_0 + \pi, \theta_0 + 3\,\pi/2$. As clearly $\kappa_n(\theta + \pi) = \kappa_n(\theta)$, and $\kappa_n(\theta)$ must achieve both maximum and minimum, there are θ_1, θ_2 such that $\theta_2 = \theta_1 + \pi/2$ and $\kappa_1 = \kappa(\theta_1) = \max \kappa(\theta)$, $\kappa_2 = \kappa(\theta_2) = \min \kappa(\theta)$. Now we can take $\tilde{e}_1 := w(\theta_1)$ and $\tilde{e}_2 := w(\theta_2)$ and re-do the calculation above using $\tilde{e}_1, \tilde{e}_2$ as the orthonomal basis and conclude that

$$H = \frac{\kappa_1 + \kappa_2}{2}. \tag{14}$$

  We call κ_1, κ_2 the *principal curvatures*, and the corresponding directions $t_1 := w(\theta_1)$, $t_2 := w(\theta_2)$ the *principal vectors* corresponding to κ_1 and κ_2.

3. How to calculate $H, K, \kappa_1, \kappa_2, t_1, t_2$.

- The calculation of Gaussian curvature is easy. Recall that

$$\begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix} = \begin{pmatrix} \mathbb{E} & \mathbb{F} \\ \mathbb{F} & \mathbb{G} \end{pmatrix}^{-1} \begin{pmatrix} \mathbb{L} & \mathbb{M} \\ \mathbb{M} & \mathbb{N} \end{pmatrix}. \tag{15}$$

  We easily obtain

$$K = \det\begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix} = \frac{\det\begin{pmatrix} \mathbb{L} & \mathbb{M} \\ \mathbb{M} & \mathbb{N} \end{pmatrix}}{\det\begin{pmatrix} \mathbb{E} & \mathbb{F} \\ \mathbb{F} & \mathbb{G} \end{pmatrix}} = \frac{\mathbb{L}\,\mathbb{N} - \mathbb{M}^2}{\mathbb{E}\,\mathbb{G} - \mathbb{F}^2}. \tag{16}$$

- For the principal and mean curvatures, we try to calculate κ_1, κ_2 in a different way. Let $w := a\,\sigma_u + b\,\sigma_v$. We try to find the maximum and minimum of

$$\kappa(w) = \mathbb{L}\,a^2 + 2\,\mathbb{M}\,a\,b + \mathbb{N}\,b^2 \tag{17}$$

under the constraint $\|w\| = 1$, that is $\mathbb{E}\,a^2 + 2\,\mathbb{F}\,a\,b + \mathbb{G}\,b^2 = 1$. To do this we apply the method of Lagrange multiplier:

$$L(a,b) := [\mathbb{L}\,a^2 + 2\,\mathbb{M}\,a\,b + \mathbb{N}\,b^2] - \lambda\,[\mathbb{E}\,a^2 + 2\,\mathbb{F}\,a\,b + \mathbb{G}\,b^2 - 1]. \tag{18}$$

Thus

$$\frac{\partial L}{\partial a} = 2\,[\mathbb{L}\,a + \mathbb{M}\,b - \lambda(\mathbb{E}\,a + \mathbb{F}\,b)], \tag{19}$$

$$\frac{\partial L}{\partial b} = 2\,[\mathbb{M}\,a + \mathbb{N}\,b - \lambda\,(\mathbb{F}\,a + \mathbb{G}\,b)]. \tag{20}$$

Setting them to zero we see that λ and $\begin{pmatrix} a \\ b \end{pmatrix}$ solves

$$\left[\begin{pmatrix} \mathbb{L} & \mathbb{M} \\ \mathbb{M} & \mathbb{N} \end{pmatrix} - \lambda \begin{pmatrix} \mathbb{E} & \mathbb{F} \\ \mathbb{F} & \mathbb{G} \end{pmatrix}\right] \begin{pmatrix} a \\ b \end{pmatrix} = 0. \tag{21}$$

This means λ solves

$$\det \begin{pmatrix} \mathbb{L} - \lambda\,\mathbb{E} & \mathbb{M} - \lambda\,\mathbb{F} \\ \mathbb{M} - \lambda\,\mathbb{F} & \mathbb{N} - \lambda\,\mathbb{G} \end{pmatrix} = 0 \tag{22}$$

which simplifies to the quadratic equation

$$(\mathbb{E}\,\mathbb{G} - \mathbb{F}^2)\,\lambda^2 - (\mathbb{E}\,\mathbb{N} + \mathbb{L}\,\mathbb{G} - 2\,\mathbb{M}\,\mathbb{F})\,\lambda + (\mathbb{L}\,\mathbb{N} - \mathbb{M}^2) = 0. \tag{23}$$

- What is λ?

We see that there are two solutions to (23). Denote them by λ_1, λ_2. Let $\begin{pmatrix} a_i \\ b_i \end{pmatrix}$ with $\mathbb{E}\,a_i^2 + 2\,\mathbb{F}\,a_i\,b_i + \mathbb{G}\,b_i^2 = 1$ solve

$$\left[\begin{pmatrix} \mathbb{L} & \mathbb{M} \\ \mathbb{M} & \mathbb{N} \end{pmatrix} - \lambda_i \begin{pmatrix} \mathbb{E} & \mathbb{F} \\ \mathbb{F} & \mathbb{G} \end{pmatrix}\right] \begin{pmatrix} a_i \\ b_i \end{pmatrix} = 0. \tag{24}$$

We see that $\kappa_n(a_i\,\sigma_u + b_i\,\sigma_v)$, for $i = 1, 2$, are the maximum and minimum of the normal curvatures. Thus

$$t_1 = a_1\,\sigma_u + b_1\,\sigma_v, \qquad t_2 = a_2\,\sigma_u + b_2\,\sigma_v \tag{25}$$

are the principal vectors.

Now we notice that (24) leads to

$$\lambda_i = \frac{\begin{pmatrix} a_i & b_i \end{pmatrix} \begin{pmatrix} \mathbb{L} & \mathbb{M} \\ \mathbb{M} & \mathbb{N} \end{pmatrix} \begin{pmatrix} a_i \\ b_i \end{pmatrix}}{\begin{pmatrix} a_i & b_i \end{pmatrix} \begin{pmatrix} \mathbb{E} & \mathbb{F} \\ \mathbb{F} & \mathbb{G} \end{pmatrix} \begin{pmatrix} a_i \\ b_i \end{pmatrix}} = \frac{\langle\langle t_i, t_i \rangle\rangle}{\langle t_i, t_i \rangle} = \kappa_i. \tag{26}$$

Thus the Lagrange multipliers are exactly the principal curvatures.

- Summarizing, we see that

$$H = \frac{\mathbb{E}\,\mathbb{N} + \mathbb{L}\,\mathbb{G} - 2\,\mathbb{M}\,\mathbb{F}}{2\,(\mathbb{E}\,\mathbb{G} - \mathbb{F}^2)}. \tag{27}$$

- An interesting consequence of the above calculation is that $K = \kappa_1\,\kappa_2$.
- Alternative characterization of κ_1, κ_2.

  Multiplying (24) from left by $\begin{pmatrix} \mathbb{E} & \mathbb{F} \\ \mathbb{F} & \mathbb{G} \end{pmatrix}^{-1}$ we see that

$$\left[\begin{pmatrix} \mathbb{E} & \mathbb{F} \\ \mathbb{F} & \mathbb{G} \end{pmatrix}^{-1} \begin{pmatrix} \mathbb{L} & \mathbb{M} \\ \mathbb{M} & \mathbb{N} \end{pmatrix} - \lambda_i \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right] \begin{pmatrix} a_i \\ b_i \end{pmatrix} = 0. \tag{28}$$

  As $\lambda_i = \kappa_i$, $\kappa_{1,2}$ are eigenvalues of the matrix $\begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix} = \begin{pmatrix} \mathbb{E} & \mathbb{F} \\ \mathbb{F} & \mathbb{G} \end{pmatrix}^{-1} \begin{pmatrix} \mathbb{L} & \mathbb{M} \\ \mathbb{M} & \mathbb{N} \end{pmatrix}$.

  Thus we have

$$H = \mathrm{Tr}\left[\begin{pmatrix} \mathbb{E} & \mathbb{F} \\ \mathbb{F} & \mathbb{G} \end{pmatrix}^{-1} \begin{pmatrix} \mathbb{L} & \mathbb{M} \\ \mathbb{M} & \mathbb{N} \end{pmatrix}\right], \qquad K = \det\left[\begin{pmatrix} \mathbb{E} & \mathbb{F} \\ \mathbb{F} & \mathbb{G} \end{pmatrix}^{-1} \begin{pmatrix} \mathbb{L} & \mathbb{M} \\ \mathbb{M} & \mathbb{N} \end{pmatrix}\right]. \tag{29}$$

Principal curvature, mean curvature, Gaussian curvature

- Principal curvatures and principal vectors.

$$\det\begin{pmatrix} \mathbb{L} - \kappa_i\,\mathbb{E} & \mathbb{M} - \kappa_i\,\mathbb{F} \\ \mathbb{M} - \kappa_i\,\mathbb{F} & \mathbb{N} - \kappa_i\,\mathbb{G} \end{pmatrix} = 0, \tag{30}$$

$$\left[\begin{pmatrix} \mathbb{L} & \mathbb{M} \\ \mathbb{M} & \mathbb{N} \end{pmatrix} - \kappa_i \begin{pmatrix} \mathbb{E} & \mathbb{F} \\ \mathbb{F} & \mathbb{G} \end{pmatrix}\right] \begin{pmatrix} a_i \\ b_i \end{pmatrix} = 0, \tag{31}$$

$$t_i = a_i\,\sigma_u + b_i\,\sigma_v, \qquad \|t_i\| = 1. \tag{32}$$

- Mean curvature.

$$H = \frac{1}{2\,\pi} \int_0^{2\pi} \kappa_n(\theta) \,\mathrm{d}\theta = \frac{\mathbb{E}\,\mathbb{N} + \mathbb{L}\,\mathbb{G} - 2\,\mathbb{M}\,\mathbb{F}}{2\,(\mathbb{E}\,\mathbb{G} - \mathbb{F}^2)} = \frac{1}{2}\,\mathrm{Tr}\left[\begin{pmatrix} \mathbb{E} & \mathbb{F} \\ \mathbb{F} & \mathbb{G} \end{pmatrix}^{-1} \begin{pmatrix} \mathbb{L} & \mathbb{M} \\ \mathbb{M} & \mathbb{N} \end{pmatrix}\right]. \tag{33}$$

- Gaussian curvature.

$$K = \lim_{r \longrightarrow 0} \frac{\text{Area of } N(B_r)}{\text{Area of } \sigma(B_r)} = \frac{\mathbb{L}\,\mathbb{N} - \mathbb{M}^2}{\mathbb{E}\,\mathbb{G} - \mathbb{F}^2} = \det\left[\begin{pmatrix} \mathbb{E} & \mathbb{F} \\ \mathbb{F} & \mathbb{G} \end{pmatrix}^{-1} \begin{pmatrix} \mathbb{L} & \mathbb{M} \\ \mathbb{M} & \mathbb{N} \end{pmatrix}\right]. \tag{34}$$

- Relations.

$$H = \frac{\kappa_1 + \kappa_2}{2}, \qquad K = \kappa_1\,\kappa_2, \qquad \kappa_{1,2} = \frac{H \pm \sqrt{H^2 - 4\,K}}{2}. \tag{35}$$

$$\kappa_n((\cos\theta)\,t_1 + (\sin\theta)\,t_2) = \kappa_1\cos^2\theta + \kappa_2\sin^2\theta. \tag{36}$$

Remark 2. We have seen last time that if $\kappa_1 = \kappa_2$ everywhere, then S is part of plane or sphere.

4. Examples

Example 3. Let $\sigma(u, v) = (u, v, f(u, v))$ be the graph of some smooth function $f(x, y)$: $U \mapsto \mathbb{R}$. Then

$$K = \frac{f_{xx}\, f_{yy} - f_{xy}^2}{(1 + f_x^2 + f_y^2)^2}, \qquad H = \frac{(1 + f_y^2)\, f_{xx} - 2\, f_x\, f_y\, f_{xy} + (1 + f_x^2)\, f_{yy}}{2\,(1 + f_x^2 + f_y^2)^{3/2}}. \tag{37}$$

Proof. We calculate

$$\sigma_u = (1, 0, f_x), \qquad \sigma_v = (0, 1, f_y), \qquad N = \frac{(-f_x, -f_y, 1)}{\sqrt{1 + f_x^2 + f_y^2}}, \tag{38}$$

$$\sigma_{uu} = (0, 0, f_{xx}), \qquad \sigma_{uv} = (0, 0, f_{xy}), \qquad \sigma_{vv} = (0, 0, f_{yy}). \tag{39}$$

Therefore

$$\mathbb{E} = 1 + f_x^2, \qquad \mathbb{F} = f_x\, f_y, \qquad \mathbb{G} = 1 + f_y^2, \tag{40}$$

$$\mathbb{L} = \frac{f_{xx}}{\sqrt{1 + f_x^2 + f_y^2}}, \qquad \mathbb{M} = \frac{f_{xy}}{\sqrt{1 + f_x^2 + f_y^2}}, \qquad \mathbb{N} = \frac{f_{yy}}{\sqrt{1 + f_x^2 + f_y^2}}. \tag{41}$$

Consequently

$$K = \frac{\mathbb{L}\,\mathbb{N} - \mathbb{M}^2}{\mathbb{E}\,\mathbb{G} - \mathbb{F}^2} = \frac{f_{xx}\, f_{yy} - f_{xy}^2}{(1 + f_x^2 + f_y^2)^2} \tag{42}$$

and

$$H = \frac{\mathbb{E}\,\mathbb{N} + \mathbb{L}\,\mathbb{G} - 2\,\mathbb{M}\,\mathbb{F}}{2\,(\mathbb{E}\,\mathbb{G} - \mathbb{F}^2)} = \frac{(1 + f_y^2)\, f_{xx} - 2\, f_x\, f_y\, f_{xy} + (1 + f_x^2)\, f_{yy}}{2\,(1 + f_x^2 + f_y^2)^{3/2}}, \tag{43}$$

as desired. □