Math 348 Differential Geometry of Curves and Surfaces

Lecture 12 The Second Fundamental Form

Xinwei Yu

Oct. 19, 2017

CAB 527, xinwei2@ualberta.ca
Department of Mathematical & Statistical Sciences
University of Alberta

Table of contents

Please do not hesitate to interrupt me if you have a question.

Review

How does a Surface Curve

- $\mathbb{L} = \sigma_{uu} \cdot N = -N_u \cdot \sigma_u$, $\mathbb{M} = \sigma_{uv} \cdot N = -N_u \cdot \sigma_v = -N_v \cdot \sigma_u$, $\mathbb{N} = \sigma_{vv} \cdot N = -N_v \cdot \sigma_v$.
- **Normal curvature and geodesic curvature.**
  - $|\kappa_n(p, w)|$[1] is the smallest possible curvature at $p \in S$ of curves passing p with its tangent at p parallel to $w \in T_pS$.
  - When $\|w\| = 1$, $\kappa_n(p, w) = \mathbb{L}w_1^2 + 2\mathbb{M}w_1w_2 + \mathbb{N}w_2^2$.
  - $\kappa N = \kappa_n N_S + \kappa_g(N_S \times T)$, $\kappa^2 = \kappa_n^2 + \kappa_g^2$.
- **Geodesic equations.**

$$\frac{\mathrm{d}}{\mathrm{d}s}(\mathbb{E}\dot{u} + \mathbb{F}\dot{v}) = \frac{1}{2}(\mathbb{E}_u\dot{u}^2 + 2\mathbb{F}_u\dot{u}\dot{v} + \mathbb{G}_u\dot{v}^2) \tag{1}$$

$$\frac{\mathrm{d}}{\mathrm{d}s}(\mathbb{F}\dot{u} + \mathbb{G}\dot{v}) = \frac{1}{2}(\mathbb{E}_v\dot{u}^2 + 2\mathbb{F}_v\dot{u}\dot{v} + \mathbb{G}_v\dot{v}^2) \tag{2}$$

when s is the arc length parameter of the curve $\gamma(s) = \sigma(u(s), v(s))$.

[1] Usually simply κ_n, also note that κ_n could be negative.

The Second Fundamental Form and Its Properties

The second fundamental form

$$\boxed{S : \sigma(u, v).\ \ p_0 = \sigma(u_0, v_0) \in S.}$$

- **Definition.** The second fundamental form of S at p_0 is a bilinear form on $T_{p_0}S$:

  $$\langle\langle v, w \rangle\rangle_{p,S} = \mathbb{L} v_1 w_1 + \mathbb{M}(v_1 w_2 + v_2 w_1) + \mathbb{N} v_2 w_2.$$

  Here $v = v_1\sigma_u + v_2\sigma_v$, $w = w_1\sigma_u + w_2\sigma_v$, and $\mathbb{L}, \mathbb{M}, \mathbb{N}$ are calculated at (u_0, v_0).
- **Classical notation.**

  $$\mathbb{L}\mathrm{d}u^2 + 2\mathbb{M}\mathrm{d}u\mathrm{d}v + \mathbb{N}\mathrm{d}v^2.$$

The Weingarten map.

- **The Weingarten map.** $\mathcal{W}_{p,S} : T_pS \mapsto T_pS.$

$$\mathcal{W}_{p,S} := -D_p\mathcal{G}.$$

  Beware of the minus sign!

- **Calculation of the Weingarten map.**

$$\mathcal{W}_{p,S}(a\sigma_u + b\sigma_v) = -aN_u - bN_v.$$

- **Matrix representation of the Weingarten map.**

$$\mathcal{W}_{p,S}(a\sigma_u + b\sigma_v) = \tilde{a}\sigma_u + \tilde{b}\sigma_v,$$

  where

$$\begin{pmatrix} \tilde{a} \\ \tilde{b} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}$$

  with

$$\begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix} = \begin{pmatrix} \mathbb{E} & \mathbb{F} \\ \mathbb{F} & \mathbb{G} \end{pmatrix}^{-1} \begin{pmatrix} \mathbb{L} & \mathbb{M} \\ \mathbb{M} & \mathbb{N} \end{pmatrix}.$$

Relations between $\langle \cdot, \cdot \rangle$ and $\langle\!\langle \cdot, \cdot \rangle\!\rangle$.

$\boxed{v, w \in T_pS\text{: Arbitrary tangent vectors.}}$

There holds

$$\langle\!\langle v, w \rangle\!\rangle_{p,S} = \langle \mathcal{W}_{p,S}(v), w \rangle_{p,S} = \langle v, \mathcal{W}_{p,S}(w) \rangle_{p,S}.$$

Proof.

1. We omit the subscripts p, S;
2. Suffices to prove for $v = \sigma_u, \sigma_v$, $w = \sigma_u, \sigma_v$;
3. Prove in detail the case $v = \sigma_u, w = \sigma_v$. Calculate

$$\begin{aligned} \langle \mathcal{W}(\sigma_u), \sigma_v \rangle &= \langle -N_u, \sigma_v \rangle \\ &= a_{11} \langle \sigma_u, \sigma_v \rangle + a_{12} \langle \sigma_v, \sigma_v \rangle \\ &= a_{11} \mathbb{F} + a_{12} \mathbb{G} \\ &= \mathbb{M} = \langle\!\langle v, w \rangle\!\rangle. \end{aligned}$$

4. Proofs for other cases are similar.

Examples

Unit sphere.

$$\boxed{\text{Keep in mind: Two formulas for each of } \mathbb{L}, \mathbb{N}, \mathbb{N}.}$$

Example

Consider the unit sphere parametrized as

1. $\sigma_1(u, v) = (u, v, \sqrt{1 - u^2 - v^2})$;
2. $\sigma_2(u, v) = (\cos u \cos v, \cos u \sin v, \sin u)$.

We can easily calculate

$$\mathbb{L}_1 = \frac{v^2 - 1}{1 - u^2 - v^2}, \quad \mathbb{M}_1 = \frac{-uv}{1 - u^2 - v^2}, \quad \mathbb{N}_1 = \frac{u^2 - 1}{1 - u^2 - v^2}.$$

and

$$\mathbb{L}_2 = -1, \qquad \mathbb{M}_2 = 0, \qquad \mathbb{N}_2 = -\cos^2 u.$$

Other surfaces.

$\boxed{\text{Keep in mind: Two formulas for each of } \mathbb{L}, \mathbb{N}, \mathbb{N}.}$

Example

$\sigma(u,v) = (u, v, u^2 + v^2)$. We easily calculate

$$\mathbb{L} = \frac{2}{\sqrt{1+4u^2+4v^2}}, \qquad \mathbb{M} = 0, \qquad \mathbb{N} = \frac{2}{\sqrt{1+4u^2+4v^2}}.$$

Example

Ruled surface: $\sigma(u,v) = \gamma(u) + v l(u)$. We see that

$$\begin{aligned} \mathbb{L} &= \|\sigma_u \times \sigma_v\|^{-1}(\ddot{\gamma} + v\ddot{l}) \cdot (\dot{\gamma} \times l + v\dot{l} \times l), \\ \mathbb{M} &= \|\sigma_u \times \sigma_v\|^{-1}\dot{l} \cdot (\dot{\gamma} \times l), \\ \mathbb{N} &= 0. \end{aligned}$$

Applications of $\langle\!\langle \cdot, \cdot \rangle\!\rangle_{p,S}$

Plane

$$\boxed{\langle\langle \cdot, \cdot \rangle\rangle_{p,S} = 0 \text{ at every } p \in S \Rightarrow \text{(part of)plane.}}$$

Proof.

1. Pick any surface patch σ for S;
2. $N_u \cdot \sigma_u = N_u \cdot \sigma_v = 0$;
3. $N_u \cdot N = 0$;
4. $N_u = 0$;
5. Similarly $N_v = 0$;
6. N is a constant vector $\Rightarrow$ S is (part of)a plane.

□

Sphere.

$$\boxed{\langle\!\langle \cdot, \cdot \rangle\!\rangle_{p,S} = \lambda(p)\langle \cdot, \cdot \rangle_{p,S} \text{ at every } p \in S \Rightarrow \text{(part of) a sphere.}}$$

Proof.

1. $\langle\!\langle \cdot, \cdot \rangle\!\rangle_{p,S} = \lambda(p)\langle \cdot, \cdot \rangle_{p,S} \Rightarrow -N_u = \lambda(u,v)\sigma_u, -N_v = \lambda(u,v)\sigma_v$;
2. Differentiate:

$$-N_{uv} = \lambda_v \sigma_u + \lambda \sigma_{uv}, \qquad -N_{vu} = \lambda_u \sigma_v + \lambda \sigma_{vu}.$$

3. $\lambda_u \sigma_v = \lambda_v \sigma_u \Rightarrow \lambda_u = \lambda_v = 0$. So $\lambda(u,v) = r$ is a constant;
4. $\sigma + \lambda N$ is a constant.

$\square$

Looking Back and Forward

Summary

- **Definitions.**
  1. Second fundamental form.

$$\langle\!\langle v, w \rangle\!\rangle_{p,S} = \mathbb{L} v_1 w_1 + \mathbb{M}(v_1 w_2 + v_2 w_1) + \mathbb{N} v_2 w_2.$$

  2. The Weingarten map. $\mathcal{W}_{p,S} := -D_p\mathcal{G}$
     $\mathcal{W}_{p,S}(a\sigma_u + b\sigma_v) = -aN_u - bN_v.$
- **Properties.**
  1. Relation between the first and second fundamental forms.

$$\langle\!\langle v, w \rangle\!\rangle_{p,S} = \langle \mathcal{W}_{p,S}(v), w \rangle_{p,S} = \langle v, \mathcal{W}_{p,S}(w) \rangle_{p,S}.$$

- **Formulas.**
  1. Matrix representation of the Weingarten map.

$$\mathcal{W}_{p,S}(a\sigma_u + b\sigma_v) = \tilde{a}\sigma_u + \tilde{b}\sigma_v, \quad \begin{pmatrix} \tilde{a} \\ \tilde{b} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}$$

     with

$$\begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix} = \begin{pmatrix} \mathbb{E} & \mathbb{F} \\ \mathbb{F} & \mathbb{G} \end{pmatrix}^{-1} \begin{pmatrix} \mathbb{L} & \mathbb{M} \\ \mathbb{M} & \mathbb{N} \end{pmatrix}.$$

See you next Tuesday!

$\boxed{\text{Curvatures for surfaces.}}$

1. Curvatures: Gaussian, mean, principal;
2. Developable surfaces;
3. Minimal surfaces.