Lectures 12: The Second Fundamental Form

Disclaimer. As we have a textbook, this lecture note is for guidance and supplement only. It should not be relied on when preparing for exams.

In this lecture we introduce the second fundamental form, its properties, and applications.

The required textbook sections are §7.1–7.3.

I try my best to make the examples in this note different from examples in the textbook. Please read the textbook carefully and try your hands on the exercises. During this please don't hesitate to contact me if you have any questions.

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1. The second fundamental form

Definition 1. (The second fundamental form) *Let S be a surface and $p_0 \in S$. Let σ be a surface patch of S covering p_0: $p_0 = \sigma(u_0, v_0)$. Then the second fundamental form of S at p_0, denoted $\langle\langle \cdot, \rangle\rangle_{p_0,S}$ (with p, S omitted when no confusion may arise), is a bilinear form on $T_{p_0}S$ defined through*

$$\langle\langle v, w\rangle\rangle_{p_0,S} = \mathbb{L}\, v_1\, w_1 + \mathbb{M}\, (v_1\, w_2 + v_2\, w_1) + \mathbb{N}\, v_2\, w_2. \tag{1}$$

where $v = v_1\, \sigma_u(u_0, v_0) + v_2\, \sigma_v(u_0, v_0)$ and $w = w_1\, \sigma_u(u_0, v_0) + w_2\, \sigma_v(u_0, v_0)$, and

$$\mathbb{L}(u_0, v_0) := \sigma_{uu}(u_0, v_0) \cdot N(u_0, v_0) = -\sigma_u \cdot N_u, \tag{2}$$

$$\mathbb{M}(u_0, v_0) := \sigma_{uv}(u_0, v_0) \cdot N(u_0, v_0) = -\sigma_u \cdot N_v = -\sigma_v \cdot N_u, \tag{3}$$

$$\mathbb{N}(u_0, v_0) := \sigma_{vv}(u_0, v_0) \cdot N(u_0, v_0) = -\sigma_v \cdot N_v. \tag{4}$$

Remark 2. An alternative notation is $\mathbb{L}(u_0, v_0)\, \mathrm{d}u^2 + 2\, \mathbb{M}(u_0, v_0)\, \mathrm{d}u\, \mathrm{d}v + \mathbb{N}(u_0, v_0)\, \mathrm{d}v^2$.

Remark 3. Let $\gamma(s) = \sigma(u(s), v(s))$ where s is the arc length parameter. If $p = \gamma(s_0)$, we clearly have

$$\kappa_n = \mathbb{L}\, (\dot{u})^2 + 2\, \mathbb{M}\, \dot{u}\, \dot{v} + \mathbb{N}\, (\dot{v})^2 = \langle\langle \dot{\gamma}, \dot{\gamma}\rangle\rangle_{\gamma(s_0),S} \tag{5}$$

We can further prove the following general formula for $w \in T_p S$

$$\kappa_n(p_0, w_0) = \frac{\langle\langle w_0, w_0\rangle\rangle_{p_0,S}}{\langle w_0, w_0\rangle_{p_0,S}}. \tag{6}$$

As a consequence, when $\gamma(t)$ is not parametrized by arc length, we have

$$\kappa_n(p_0, \dot{\gamma}(t_0)) = \frac{\langle\langle \dot{\gamma}, \dot{\gamma}\rangle\rangle_{\gamma(t_0),S}}{\langle \dot{\gamma}, \dot{\gamma}\rangle_{\gamma(t_0),S}} = \frac{\mathbb{L}\, (\dot{u})^2 + 2\, \mathbb{M}\, \dot{u}\, \dot{v} + \mathbb{N}\, (\dot{v})^2}{\mathbb{E}\, (\dot{u})^2 + 2\, \mathbb{F}\, \dot{u}\, \dot{v} + \mathbb{G}\, (\dot{v})^2}. \tag{7}$$

2. The Weingarten map

Definition 4. (Definition 7.2.1 in the textbook) *We define the* Weingarten map

$$\mathcal{W}_{p_0,S} := -D_{p_0}\mathcal{G} \tag{8}$$

where $\mathcal{G}$ is the Gauss map.

Note the minus sign here.

Example 5. We try to calculate $\mathcal{W}_{p_0,S}(\sigma_u)$ and $\mathcal{W}_{p_0,S}(\sigma_v)$ for the following surface patches. It is clear that

$$\mathcal{W}_{p_0,S}(\sigma_u) = -N_u, \qquad \mathcal{W}_{p_0,S}(\sigma_v) = -N_v. \tag{9}$$

a) S is the plane $\sigma(u, v) = (u, v, 3\, u + 2\, v)$.

In this case we have

$$\sigma_u = (1, 0, 3), \qquad \sigma_v = (0, 1, 2) \tag{10}$$

which give

$$N(u, v) = \mathcal{G}(\sigma(u, v)) = \frac{\sigma_u \times \sigma_v}{\|\sigma_u \times \sigma_v\|} = \frac{1}{\sqrt{14}}(-3, -2, 1). \tag{11}$$

We see that $\mathcal{W}(\sigma_u) = \mathcal{W}(\sigma_v) = 0$.

b) S is the cylinder $\sigma(u, v) = (\cos u, \sin u, v)$.
In this case we have

$$\sigma_u = (-\sin u, \cos u, 0), \qquad \sigma_v = (0, 0, 1) \tag{12}$$

and

$$N(u, v) = \frac{\sigma_u \times \sigma_v}{\|\sigma_u \times \sigma_v\|} = (\cos u, \sin u, 0). \tag{13}$$

We have

$$N_u = (-\sin u, \cos u, 0) = \sigma_u, \qquad N_v = (0, 0, 0). \tag{14}$$

Consequently we have

$$\mathcal{W}(\sigma_u) = -\sigma_u, \qquad \mathcal{W}(\sigma_v) = 0. \tag{15}$$

c) S is the unit sphere $\sigma(u, v) = \left(u, v, \sqrt{1 - u^2 - v^2}\right)$.
We have

$$\sigma_u = \left(1, 0, \frac{-u}{\sqrt{1 - u^2 - v^2}}\right), \qquad \sigma_v = \left(0, 1, \frac{-v}{\sqrt{1 - u^2 - v^2}}\right) \tag{16}$$

and

$$N(u, v) = \left(u, v, \sqrt{1 - u^2 - v^2}\right) = \sigma(u, v). \tag{17}$$

Consequently

$$\mathcal{W}(\sigma_u) = -N_u, \qquad \mathcal{W}(\sigma_v) = -N_v. \tag{18}$$

d) S is the hyperbolic paraboloid $\sigma(u, v) = (u, v, u\,v)$ with $p_0 = (0, 0, 0)$.
We have

$$\sigma_u = (1, 0, v), \qquad \sigma_v = (0, 1, u) \tag{19}$$

and

$$N(u, v) = \left(\frac{-v}{\sqrt{1 + u^2 + v^2}}, \frac{-u}{\sqrt{1 + u^2 + v^2}}, \frac{1}{\sqrt{1 + u^2 + v^2}}\right). \tag{20}$$

Now we calculate

$$\mathcal{W}(\sigma_u) = -N_u = \left(\frac{-u\,v}{(1 + u^2 + v^2)^{3/2}}, \frac{1 + v^2}{(1 + u^2 + v^2)^{3/2}}, \frac{u}{(1 + u^2 + v^2)^{3/2}}\right) \tag{21}$$

and

$$\mathcal{W}(\sigma_v) = -N_v = \left(\frac{1 + u^2}{(1 + u^2 + v^2)^{3/2}}, \frac{-u\,v}{(1 + u^2 + v^2)^{3/2}}, \frac{v}{(1 + u^2 + v^2)^{3/2}}\right). \tag{22}$$

We see that

$$\mathcal{W}(\sigma_u) = -\frac{u\,v}{(1+u^2+v^2)^{3/2}}\,\sigma_u + \frac{1+v^2}{(1+u^2+v^2)^{3/2}}\,\sigma_v \tag{23}$$

and

$$\mathcal{W}(\sigma_v) = \frac{1+u^2}{(1+u^2+v^2)^{3/2}}\,\sigma_u - \frac{u\,v}{(1+u^2+v^2)^{3/2}}\,\sigma_v. \tag{24}$$

We have seen that $\mathcal{W}(\sigma_u) = -N_u$, $\mathcal{W}(\sigma_v) = -N_v$. As $\mathcal{W}$ is linear, for $a, b \in \mathbb{R}$ we have

$$\mathcal{W}(a\,\sigma_u + b\,\sigma_v) = -a\,N_u - b\,N_v. \tag{25}$$

Therefore to understand $\mathcal{W}$ we need to understand N_u, N_v. The crucial observation is the following.

$$N_u, N_v \perp N \Longrightarrow -N_u = a_{11}\,\sigma_u + a_{12}\,\sigma_v, -N_v = a_{21}\,\sigma_u + a_{22}\,\sigma_v.$$

Theorem 6. *We have*

$$\begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix} = \begin{pmatrix} \mathbb{E} & \mathbb{F} \\ \mathbb{F} & \mathbb{G} \end{pmatrix}^{-1} \begin{pmatrix} \mathbb{L} & \mathbb{M} \\ \mathbb{M} & \mathbb{N} \end{pmatrix} \tag{26}$$

where $\mathbb{E}\,\mathrm{d}u^2 + 2\,\mathbb{F}\,\mathrm{d}u\,\mathrm{d}v + \mathbb{G}\,\mathrm{d}v^2$ *is the first fundamental form of* S *at* p_0*, and* $\mathbb{L}, \mathbb{M}, \mathbb{N}$ *are defined in (4–6).*

Proof. We notice that as $\sigma_u \cdot N = \sigma_v \cdot N = 0$, there holds

$$\mathbb{L} = \sigma_{uu} \cdot N = (\sigma_u \cdot N)_u - \sigma_u \cdot N_u = -\sigma_u \cdot N_u \tag{27}$$

and similarly

$$\mathbb{M} = -\sigma_v \cdot N_u = -\sigma_u \cdot N_v, \qquad \mathbb{N} = -\sigma_v \cdot N_v. \tag{28}$$

This leads to

$$\mathbb{E}\,a_{11} + \mathbb{F}\,a_{12} = \sigma_u \cdot (a_{11}\,\sigma_u + a_{12}\,\sigma_v) = -\sigma_u \cdot N_u = \mathbb{L}, \tag{29}$$

$$\mathbb{F}\,a_{11} + \mathbb{G}\,a_{12} = \sigma_v \cdot (a_{11}\,\sigma_u + a_{12}\,\sigma_v) = -\sigma_v \cdot N_u = \mathbb{M}. \tag{30}$$

Consequently

$$\begin{pmatrix} a_{11} \\ a_{12} \end{pmatrix} = \begin{pmatrix} \mathbb{E} & \mathbb{F} \\ \mathbb{F} & \mathbb{G} \end{pmatrix}^{-1} \begin{pmatrix} \mathbb{L} \\ \mathbb{M} \end{pmatrix}. \tag{31}$$

Similarly we have $\begin{pmatrix} a_{21} \\ a_{22} \end{pmatrix} = \begin{pmatrix} \mathbb{E} & \mathbb{F} \\ \mathbb{F} & \mathbb{G} \end{pmatrix}^{-1} \begin{pmatrix} \mathbb{M} \\ \mathbb{N} \end{pmatrix}$ and the conclusion follows. $\square$

Lemma 7. *Let* $v, w \in T_pS$*. Then*

$$\langle\langle v, w \rangle\rangle_{p,S} = \langle \mathcal{W}_{p,S}(v), w \rangle_{p,S} = \langle v, \mathcal{W}_{p,S}(w) \rangle_{p,S}. \tag{32}$$

Proof. Since $\langle\langle \cdot, \cdot \rangle\rangle_{p,S}$, $\langle \mathcal{W}_{p,S}(\cdot), \cdot \rangle_{p,S}$, and $\langle \cdot, \mathcal{W}_{p,S}(\cdot) \rangle_{p,S}$ are all bilinear, it suffices to prove the following cases: $v = \sigma_u, w = \sigma_v$; $v = w = \sigma_u$; $v = w = \sigma_v$; $v = \sigma_v, w = \sigma_u$. We prove the first one and leave the other three as exercises.

We calculate

$$\langle\langle\sigma_u, \sigma_v\rangle\rangle_{p,S} = \mathbb{M}. \tag{33}$$

On the other hand, $\mathcal{W}_{p,S}(\sigma_u) = -N_u = a_{11}\sigma_u + a_{12}\sigma_v$ where

$$\begin{pmatrix} a_{11} \\ a_{12} \end{pmatrix} = \begin{pmatrix} \mathbb{E} & \mathbb{F} \\ \mathbb{F} & \mathbb{G} \end{pmatrix}^{-1} \begin{pmatrix} \mathbb{L} \\ \mathbb{M} \end{pmatrix}. \tag{34}$$

Consequently

$$\begin{aligned} \langle\mathcal{W}_{p,S}(\sigma_u), \sigma_v\rangle_{p,S} &= a_{11}\langle\sigma_u, \sigma_v\rangle_{p,S} + a_{12}\langle\sigma_v, \sigma_v\rangle_{p,S} \\ &= a_{11}\mathbb{F} + a_{12}\mathbb{G} \\ &= \begin{pmatrix} \mathbb{F} & \mathbb{G} \end{pmatrix} \begin{pmatrix} \mathbb{E} & \mathbb{F} \\ \mathbb{F} & \mathbb{G} \end{pmatrix}^{-1} \begin{pmatrix} \mathbb{L} \\ \mathbb{M} \end{pmatrix} \\ &= \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbb{L} \\ \mathbb{M} \end{pmatrix} = \mathbb{M}. \end{aligned} \tag{35}$$

Note that we have used

$$\begin{pmatrix} \mathbb{E} & \mathbb{F} \\ \mathbb{F} & \mathbb{G} \end{pmatrix} \begin{pmatrix} \mathbb{E} & \mathbb{F} \\ \mathbb{F} & \mathbb{G} \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \Longrightarrow \begin{pmatrix} \mathbb{F} & \mathbb{G} \end{pmatrix} \begin{pmatrix} \mathbb{E} & \mathbb{F} \\ \mathbb{F} & \mathbb{G} \end{pmatrix}^{-1} = \begin{pmatrix} 0 & 1 \end{pmatrix}. \tag{36}$$

The proof that $\langle\sigma_u, \mathcal{W}_{p,S}(\sigma_v)\rangle_{p,S} = \mathbb{M}$ is similar. □

3. Examples

Example 8. Consider the unit sphere $\left(u, v, \sqrt{1-u^2-v^2}\right)$. We calculate

$$\sigma_u = \left(1, 0, \frac{-u}{\sqrt{1-u^2-v^2}}\right), \qquad \sigma_v = \left(0, 1, \frac{-v}{\sqrt{1-u^2-v^2}}\right) \tag{37}$$

which gives

$$N(u,v) = \frac{\sigma_u \times \sigma_v}{\|\sigma_u \times \sigma_v\|} = \left(u, v, \sqrt{1-u^2-v^2}\right) = \sigma(u,v). \tag{38}$$

Therefore

$$\mathbb{L}(u,v) = -\sigma_u \cdot N_u = \frac{v^2-1}{1-u^2-v^2}, \tag{39}$$

$$\mathbb{M}(u,v) = -\sigma_u \cdot N_v = \frac{-u\,v}{1-u^2-v^2}, \tag{40}$$

$$\mathbb{N}(u,v) = -\sigma_v \cdot N_v = \frac{u^2-1}{1-u^2-v^2}. \tag{41}$$

Example 9. Consider the unit sphere in spherical coordinates $(\cos u \cos v, \cos u \sin v, \sin u)$. We calculate

$$\sigma_u = (-\sin u \cos v, -\sin u \sin v, \cos u), \qquad \sigma_v = (-\cos u \sin v, \cos u \cos v, 0) \tag{42}$$

which gives

$$N(u,v) = (\cos u \cos v, \cos u \sin v, \sin u). \tag{43}$$

Therefore

$$\begin{aligned}\mathbb{L}(u,v) &= -1, &(44)\\ \mathbb{M}(u,v) &= 0, &(45)\\ \mathbb{N}(u,v) &= -\cos^2 u. &(46)\end{aligned}$$

Example 10. Consider the surface patch $\sigma(u,v) = (u, v, u^2 + v^2)$. We have

$$\sigma_u = (1, 0, 2\,u), \qquad \sigma_v = (0, 1, 2\,v), \tag{47}$$

$$\sigma_{uu} = \sigma_{vv} = (0, 0, 2), \qquad \sigma_{uv} = (0, 0, 0), \tag{48}$$

and

$$N = \frac{\sigma_u \times \sigma_v}{\|\sigma_u \times \sigma_v\|} = \frac{(-2\,u, -2\,v, 1)}{\sqrt{1 + 4\,u^2 + 4\,v^2}}. \tag{49}$$

Thus we have

$$\mathbb{L} = \sigma_{uu} \cdot N = \frac{2}{\sqrt{1 + 4\,u^2 + 4\,v^2}}, \tag{50}$$

$$\mathbb{M} = \sigma_{uv} \cdot N = 0, \tag{51}$$

$$\mathbb{N} = \sigma_{vv} \cdot N = \frac{2}{\sqrt{1 + 4\,u^2 + 4\,v^2}}. \tag{52}$$

So the second fundamental form is

$$\frac{2}{\sqrt{1 + 4\,u^2 + 4\,v^2}}\,(\mathrm{d}u^2 + \mathrm{d}v^2). \tag{53}$$

Exercise 1. Does this mean at any point $p \in S$, the normal curvature κ_n is a constant in every direction?

Example 11. Consider a ruled surface $\sigma(u,v) = \gamma(u) + v\,l(u)$ where $l(u)$ is of unit length. We calculate

$$\sigma_u = \dot{\gamma}(u) + v\,\dot{l}(u), \qquad \sigma_v = l(u). \tag{54}$$

This gives

$$N(u,v) = \frac{\sigma_u \times \sigma_v}{\|\sigma_u \times \sigma_v\|} = \frac{\dot{\gamma}(u) \times l(u) + v\,\dot{l}(u) \times l(u)}{\left\|\dot{\gamma}(u) \times l(u) + v\,\dot{l}(u) \times l(u)\right\|}. \tag{55}$$

We further calculate

$$\sigma_{uu} = \ddot{\gamma}(u) + v\,\ddot{l}(u), \quad \sigma_{uv} = \dot{l}(u), \quad \sigma_{vv} = 0. \tag{56}$$

Therefore if we set $A = \|\sigma_u \times \sigma_v\|$.

$$\begin{aligned}\mathbb{L}(u,v) &= \sigma_{uu} \cdot N = A^{-1}\left(\ddot{\gamma} + v\,\ddot{l}\right) \cdot \left(\dot{\gamma}(u) \times l(u) + v\,\dot{l}(u) \times l(u)\right), &(57)\\ \mathbb{M}(u,v) &= \sigma_{uv} \cdot N = A^{-1}\,\dot{l} \cdot (\dot{\gamma} \times l), &(58)\\ \mathbb{N}(u,v) &= \sigma_{vv} \cdot N = 0. &(59)\end{aligned}$$

Recalling our discussion on developable surfaces, we see that a ruled surface is developable if and only if $\mathbb{M} = 0$.

4. Applications of the second fundamental form

PROPOSITION 12. *Let S be a surface whose second fundamental form is identically zero. Then S is part of a plane.*

Proof. Let σ be a surface patch for S. Then by assumption we have $N_u \cdot \sigma_u = N_u \cdot \sigma_v = 0$. As N is the unit normal, naturally $N_u \cdot N = 0$. Consequently $N_u = 0$ as $\{\sigma_u, \sigma_v, N\}$ form a basis of $\mathbb{R}^3$. Similarly $N_v = 0$. Thus N is a constant vector and therefore σ is part of a plane. $\square$

PROPOSITION 13. *Let S be a suface whose second fundamental form at every $p \in S$ is a non-zero scalar multiple of its first fundamental form at p. Then S is part of a sphere.*

Exercise 2. Prove that if S is part of a sphere, then its second fundamental form is a non-zero scalar multiple of its first fundamental form.

Proof. Let $\sigma(u,v)$ be a surface patch for S. Then there holds

$$\mathbb{L}(u,v) = c(u,v)\,\mathbb{E}(u,v), \qquad \mathbb{M}(u,v) = c(u,v)\,\mathbb{F}(u,v), \qquad \mathbb{N}(u,v) = c(u,v)\,\mathbb{G}(u,v) \tag{60}$$

for every (u,v). This leads to

$$\begin{pmatrix} \mathbb{E} & \mathbb{F} \\ \mathbb{F} & \mathbb{G} \end{pmatrix}^{-1} \begin{pmatrix} \mathbb{L} & \mathbb{M} \\ \mathbb{M} & \mathbb{N} \end{pmatrix} = c(u,v) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{61}$$

As a consequence, we have

$$N_u + c(u,v)\,\sigma_u = 0, \qquad N_v + c(u,v)\,\sigma_v = 0 \tag{62}$$

at every (u,v). Taking v, u derivatives of the two equations respectively, we have

$$N_{uv} + c_v\,\sigma_u + c\,\sigma_{uv} = 0 = N_{vu} + c_u\,\sigma_v + c\,\sigma_{vu} \Longrightarrow c_v\,\sigma_u = c_u\,\sigma_v. \tag{63}$$

As σ_u, σ_v form a basis of T_pS, there must hold $c_v = c_u = 0$, that is $c(u,v) = c$ is a constant.

Now (62) becomes

$$(N + c\,\sigma)_u = (N + c\,\sigma)_v = 0 \Longrightarrow N + c\,\sigma = r_0 \tag{64}$$

is a constant. In other words, we have

$$\sigma + c^{-1} N = c^{-1} r_0 \tag{65}$$

is a constant which means σ is part of the sphere centered at $c^{-1} r_0$ and with radius $|c|^{-1}$. $\square$