Math 348 Differential Geometry of Curves and Surfaces

Lecture 11 The Second Fundamental Form: Measuring How a Surface Curves

Xinwei Yu

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CAB 527, xinwei2@ualberta.ca
Department of Mathematical & Statistical Sciences
University of Alberta

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Please do not hesitate to interrupt me if you have a question.

Review

The First Fundamental Form

$$\boxed{\mathbb{E} = \sigma_u \cdot \sigma_u, \mathbb{F} = \sigma_u \cdot \sigma_v, \mathbb{G} = \sigma_v \cdot \sigma_v.}$$

- **Intuition.**

  Different "scale" at different points on the map.

- **Definition.** At each $p \in S$, a bilinear form on T_pS.

  $v, w \in T_pS : v = v_1\sigma_u + v_2\sigma_v, w = w_1\sigma_u + w_2\sigma_v.$

  $$\langle v, w \rangle_{p,S} = \mathbb{E}v_1w_1 + \mathbb{F}(v_1w_2 + v_2w_1) + \mathbb{G}v_2w_2$$

- **Measurements.**
  1. Arc length: $L = \int_a^b \langle \dot\gamma, \dot\gamma \rangle_{p,S}^{1/2} \mathrm{d}t$;
  2. Angle: $\cos \angle(v, w) = \frac{\langle v,w \rangle_{p,S}}{\langle v,v \rangle_{p,S}^{1/2} \langle w,w \rangle_{p,S}^{1/2}}$;
  3. Area: $A = \int_U \sqrt{\mathbb{E}\mathbb{G} - \mathbb{F}^2} \mathrm{d}u\mathrm{d}v$.

- **Properties.**
  - Isometry: Preserves arc length. $\mathbb{E}_1 = \mathbb{E}_2,\ \mathbb{F}_1 = \mathbb{F}_2,\ \mathbb{G}_1 = \mathbb{G}_2$.
  - Conformal: Preserves angle. $\mathbb{E}_1 = \lambda\mathbb{E}_2,\ \mathbb{F}_1 = \lambda\mathbb{F}_2,\ \mathbb{G}_1 = \lambda\mathbb{G}_2$.
  - Equiareal: Preserves area. $\mathbb{E}_1\mathbb{G}_1 - \mathbb{F}_1^2 = \mathbb{E}_2\mathbb{G}_2 - \mathbb{F}_2^2$.

Curving of a Surface

How to measure?

$$\boxed{S : \sigma(u, v).\ p_0 = \sigma(u_0, v_0) \in S. \text{ How does } S \text{ curve at } p_0?}$$

1. How quickly does S deviate from T_pS?
2. How quickly does N turn?
3. Are curves passing p "forced" to curve?

Deviation from T_pS.

- **The tangent plane.**

$$T_{p_0}S: \quad (x - p_0)\cdot N_{p_0} = 0.$$

- **Distance to $T_{p_0}S$.**

$$d = \left|(\sigma(u,v) - p_0)\cdot N(u_0,v_0)\right|.$$

- $\mathbb{L}, \mathbb{M}, \mathbb{N}$.

$$\begin{aligned}
& (\sigma(u,v) - p_0)\cdot N(u_0,v_0) \\
= \quad & (\sigma(u,v) - \sigma(u_0,v_0))\cdot N(u_0,v_0) \\
= \quad & [\sigma_u(u-u_0) + \sigma_v(v-v_0)]\cdot N(u_0,v_0) + \\
& \left[\frac{1}{2}\sigma_{uu}(u-u_0)^2 + \sigma_{uv}(u-u_0)(v-v_0) + \frac{1}{2}\sigma_{vv}(v-v_0)^2\right]\cdot N(u_0,v_0) \\
& + h.o.t. \\
= \quad & \frac{1}{2}\left[\mathbb{L}(u-u_0)^2 + 2\mathbb{M}(u-u_0)(v-v_0) + \mathbb{N}(v-v_0)^2\right] + h.o.t.
\end{aligned}$$

$$\boxed{\mathbb{L} = \sigma_{uu}\cdot N, \quad \mathbb{M} = \sigma_{uv}\cdot N, \quad \mathbb{N} = \sigma_{vv}\cdot N.}$$

Turning of the unit normal

$\boxed{\text{How does } N_p \text{ change along the surface?}}$

- **Change of N.**

$$N_u, N_v.$$

- **How to measure N_u, N_v?**
  - $N_u, N_v \perp N \Rightarrow N_u, N_v \in T_{p_0}S$;
  - $N_u, N_v \in span \sigma_u, \sigma_v$;
  - Solve

$$-N_u = a_{11}\sigma_u + a_{12}\sigma_v, \qquad -N_v = a_{21}\sigma_u + a_{22}\sigma_v.$$

  - We have

$$\mathbb{L} = -N_u \cdot \sigma_u, \quad \mathbb{M} = -N_u \cdot \sigma_v = -N_v \cdot \sigma_u, \quad \mathbb{N} = -N_v \cdot \sigma_v.$$

  - Obtain

$$\begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix} = \begin{pmatrix} \mathbb{E} & \mathbb{F} \\ \mathbb{F} & \mathbb{G} \end{pmatrix}^{-1} \begin{pmatrix} \mathbb{L} & \mathbb{M} \\ \mathbb{M} & \mathbb{N} \end{pmatrix}$$

Example

Consider curves on

1. a plane;
2. a cylinder;
3. a sphere.

in any direction, or in a fixed direction.

How are curves forced to curve?

$$\text{Fix } p_0 \in S,\ w_0 \in T_{p_0}S.$$

$$\boxed{\text{Consider infimum of curvatures of curves passing } p_0 \text{ in direction } w_0.}$$

- **Curve in S.** $\gamma(s) = \sigma(u(s), v(s))$. Let s be the arc length parameter.
- **Notation.** Let T, N, B be the tangent, normal, binormal of γ, let N_S denote the unit normal of S.
- **Fix point, fix direction.**

$$u(s_0) = u_0, v(s_0) = v_0; \qquad \dot{u}(s_0) = u_1, \dot{v}(s_0) = v_1.$$

- **Calculation of κ.**

$$\ddot{\gamma} = \ddot{u}\sigma_u + \ddot{v}\sigma_v + u_1^2\sigma_{uu} + 2u_1v_1\sigma_{uv} + v_1^2\sigma_{vv}.$$

$$\Rightarrow \kappa \geqslant |\ddot{\gamma} \cdot N_S| = \left|\mathbb{L}u_1^2 + 2\mathbb{M}u_1v_1 + \mathbb{N}v_1^2\right|.$$

Infimum or minimum?

$$\boxed{\kappa \geqslant |\ddot{\gamma} \cdot N_S| = \left|\mathbb{L} u_1^2 + 2\mathbb{M} u_1 v_1 + \mathbb{N} v_1^2\right|.}$$

- **Minimum if**

$$\ddot{\gamma} \cdot \sigma_u = \ddot{\gamma} \cdot \sigma_v = 0.$$

- **Nonlinear ODE system.**

$$\frac{\mathrm{d}}{\mathrm{d}s}(\mathbb{E}\dot{u} + \mathbb{F}\dot{v}) = \frac{1}{2}(\mathbb{E}_u \dot{u}^2 + 2\mathbb{F}_u \dot{u}\dot{v} + \mathbb{G}_u \dot{v}^2) \tag{1}$$

$$\frac{\mathrm{d}}{\mathrm{d}s}(\mathbb{F}\dot{u} + \mathbb{G}\dot{v}) = \frac{1}{2}(\mathbb{E}_v \dot{u}^2 + 2\mathbb{F}_v \dot{u}\dot{v} + \mathbb{G}_v \dot{v}^2) \tag{2}$$

Must be in arc length parametrization!

- **ODE existence theory $\Rightarrow$ minimum.**

Will see: Such curves are geodesics.

Normal and geodesic curvatures.

$$\boxed{\kappa N = \kappa_n N_S + \kappa_g (N_S \times T).}$$

- κ_n is a property of the surface; κ_g relies on details of the curve.
- $$\kappa^2 = \kappa_n^2 + \kappa_g^2.$$
- $\kappa_n(p, w)$ is the smallest possible curvature of a curve passing p in direction w.

Looking Back and Forward

Summary

Required: §7.1–7.3; Optional:

- **Definitions.**
  1. $\mathbb{L} = \sigma_{uu} \cdot N, \quad \mathbb{M} = \sigma_{uv} \cdot N, \quad \mathbb{N} = \sigma_{vv} \cdot N.$
  2. $\kappa N = \kappa_n N_S + \kappa_g (N_S \times T).$
- **Properties.**
  1. $\kappa_n(p, w)$ is the smallest possible curvature of a curve passing p in direction w.
  2. $\mathbb{L} = -N_u \cdot \sigma_u, \quad \mathbb{M} = -N_u \cdot \sigma_v = -N_v \cdot \sigma_u, \quad \mathbb{N} = -N_v \cdot \sigma_v.$
- **Equations.**
  1. Geodesic equations.

$$\frac{\mathrm{d}}{\mathrm{d}s}(\mathbb{E}\dot{u} + \mathbb{F}\dot{v}) = \frac{1}{2}(\mathbb{E}_u \dot{u}^2 + 2\mathbb{F}_u \dot{u}\dot{v} + \mathbb{G}_u \dot{v}^2) \tag{3}$$

$$\frac{\mathrm{d}}{\mathrm{d}s}(\mathbb{F}\dot{u} + \mathbb{G}\dot{v}) = \frac{1}{2}(\mathbb{E}_v \dot{u}^2 + 2\mathbb{F}_v \dot{u}\dot{v} + \mathbb{G}_v \dot{v}^2) \tag{4}$$

See you next Tuesday!

$\boxed{\text{The second fundamental form } \langle\!\langle \cdot, \cdot \rangle\!\rangle.}$

1. Definition of the second fundamental form;
2. Properties;
3. Examples;
4. Use the second fundamental form to understand surfaces.