Lectures 11: How Does a Surface Curve

Disclaimer. As we have a textbook, this lecture note is for guidance and supplement only. It should not be relied on when preparing for exams.

In this lecture we study how to measure the curving of a surface patch. The required textbook sections are §7.1–7.3.

I try my best to make the examples in this note different from examples in the textbook. Please read the textbook carefully and try your hands on the exercises. During this please don't hesitate to contact me if you have any questions.

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Let S be a surface and let $p_0 \in S$. Let $\sigma: U \mapsto \mathbb{R}^3$ be a surface patch covering p_0. Let $\sigma(u_0, v_0) = p_0$. In the following we study three ways to measure how the surface curves at p_0.

1. Distance to the tangent plane

- We measure the curving of the surface by calculating how quickly the surface curves away from its tangent plane at p_0. Note that the tangent plane is the best flat approximation of the surface that passes p_0.
- Recall that the equation for the tangent plane in $\mathbb{R}^3$ is given by

$$(x - p_0) \cdot N(p_0) = 0. \tag{1}$$

- Let $p = \sigma(u, v) \in S$ be a point close to p_0. Then we have its distance to the tangent plane to be

$$d(u, v) = |(\sigma(u, v) - \sigma(u_0, v_0)) \cdot N(\sigma(u_0, v_0))|. \tag{2}$$

- We calculate $d(u, v)$ through Taylor expansion:

$$\begin{aligned}(\sigma(u,v) - \sigma(u_0, v_0)) \cdot N(\sigma(u_0, v_0)) &= [\sigma_u (u - u_0) + \sigma_v (v - v_0)] \cdot N \\ &\quad + \left[\frac{1}{2} \sigma_{uu} (u - u_0)^2 + \sigma_{uv} (u - u_0)(v - v_0) + \frac{1}{2} \sigma_{vv} (v - v_0)^2\right] \cdot N + R(u,v) \cdot N \\ &= \frac{1}{2} [\mathbb{L} (u - u_0)^2 + 2\, \mathbb{M} (u - u_0)(v - v_0) + \mathbb{N} (v - v_0)^2] + R(u,v) \cdot N, \end{aligned} \tag{3}$$

where $\lim_{(u,v) \longrightarrow (u_0,v_0)} \frac{|R(u,v)|}{(u - u_0)^2 + (v - v_0)^2} = 0$.

- Thus we see that the curving of the surface at p_0 can be characterized by three numbers:

$$\mathbb{L}(u_0, v_0) := \sigma_{uu}(u_0, v_0) \cdot N(u_0, v_0), \tag{4}$$

$$\mathbb{M}(u_0, v_0) := \sigma_{uv}(u_0, v_0) \cdot N(u_0, v_0), \tag{5}$$

$$\mathbb{N}(u_0, v_0) := \sigma_{vv}(u_0, v_0) \cdot N(u_0, v_0). \tag{6}$$

Exercise 1. Would we obtain the same numbers if we use $N(\sigma(u, v))$ instead of $N(\sigma(u_0, v_0))$ in (2)?

2. The turning of the unit normal

- Recall that the unit normal vector $N(p) := \frac{\sigma_u \times \sigma_v}{\|\sigma_u \times \sigma_v\|}$
- The crucial observation is the following.

$$N_u, N_v \perp N \Longrightarrow -N_u = a_{11} \sigma_u + a_{12} \sigma_v, -N_v = a_{21} \sigma_u + a_{22} \sigma_v.$$

- Calculating $a_{11}, \ldots, a_{22}$.

THEOREM 1. *We have*

$$\begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix} = \begin{pmatrix} \mathbb{E} & \mathbb{F} \\ \mathbb{F} & \mathbb{G} \end{pmatrix}^{-1} \begin{pmatrix} \mathbb{L} & \mathbb{M} \\ \mathbb{M} & \mathbb{N} \end{pmatrix} \tag{7}$$

where $\mathbb{E}\,\mathrm{d}u^2 + 2\,\mathbb{F}\,\mathrm{d}u\,\mathrm{d}v + \mathbb{G}\,\mathrm{d}v^2$ *is the first fundamental form of* S *at* p_0*, and* $\mathbb{L}, \mathbb{M}$, $\mathbb{N}$ *are defined in (4–6).*

Proof. We notice that as $\sigma_u \cdot N = \sigma_v \cdot N = 0$, there holds

$$\mathbb{L} = \sigma_{uu} \cdot N = (\sigma_u \cdot N)_u - \sigma_u \cdot N_u = -\sigma_u \cdot N_u \tag{8}$$

and similarly

$$\mathbb{M} = -\sigma_v \cdot N_u = -\sigma_u \cdot N_v, \qquad \mathbb{N} = -\sigma_v \cdot N_v. \tag{9}$$

This leads to

$$\mathbb{E}\,a_{11} + \mathbb{F}\,a_{12} = \sigma_u \cdot (a_{11}\sigma_u + a_{12}\sigma_v) = -\sigma_u \cdot N_u = \mathbb{L}, \tag{10}$$

$$\mathbb{F}\,a_{11} + \mathbb{G}\,a_{12} = \sigma_v \cdot (a_{11}\sigma_u + a_{12}\sigma_v) = -\sigma_v \cdot N_u = \mathbb{M}. \tag{11}$$

Consequently

$$\begin{pmatrix} a_{11} \\ a_{12} \end{pmatrix} = \begin{pmatrix} \mathbb{E} & \mathbb{F} \\ \mathbb{F} & \mathbb{G} \end{pmatrix}^{-1} \begin{pmatrix} \mathbb{L} \\ \mathbb{M} \end{pmatrix}. \tag{12}$$

Similarly we have $\begin{pmatrix} a_{21} \\ a_{22} \end{pmatrix} = \begin{pmatrix} \mathbb{E} & \mathbb{F} \\ \mathbb{F} & \mathbb{G} \end{pmatrix}^{-1} \begin{pmatrix} \mathbb{M} \\ \mathbb{N} \end{pmatrix}$ and the conclusion follows. $\square$

3. How much are the curves in the surface forced to curve?

- Let $\gamma(t) := \sigma(u(t), v(t))$ be a curve in S with $u(t_0) = u_0$, $v(t_0) = v_0$. Thus it passes $p_0 = \sigma(u_0, v_0)$. We try to understand the curving of S at p_0 through the curvature of $\gamma(t)$ at $\gamma(t_0)$.
- To make this idea work we need to first qualitatively understand how are the curving of S at p_0 and the curvature of $\gamma(t)$ related.

Example 2. We consider the following paradigm situations.

- ○ Let S be the plane and $p_0 \in S$. Clearly a curve passing p_0 can have any curvature.
- ○ Let S be the cylinder and $p_0 \in S$. Again a curve passing p_0 can have arbitrary $\kappa_0 \geqslant 0$ as its curvature there.
- ○ Let S be the unit sphere. Intuitively we see that a curve passing $p_0 \in S$ could have any curvature $\geqslant 1$ but not < 1.

Exercise 2. Prove this.

From these examples it seems that the relations between the curvature of $\gamma(t)$ and the curving S is very loose. However, this relation becomes much more precise when we consider not all possible curvatures, but the minimal one:

> Given any unit vector $w_0 \in T_{p_0}S$, let $\kappa_{\min}(w_0)$ be the minimal curvature of all possible curvatures of the curves passing p_0 and are tangent to w_0 at p_0.

Now we see that $\kappa_{\min}$ very precisely reflects the curving of the surface.

- For S the flat plane: $\kappa_{\min}(w_0) = 0$ for all w_0;
- For S the cylinder: $\kappa_{\min}(w_0) = 0$ when $w_0 = (0, 0, 1)$ and $\kappa_{\min}(w_0) = 1$ when w_0 is the horizontal tangent, and $\kappa_{\min}(w_0)$ lies between 0 and 1 for other directions.
- For S the sphere: $\kappa_{\min}(w_0) = 1$ for all w_0.

- What is $\kappa_{\min}(w_0)$?

First we re-parametrize by arc length $\gamma(s) = \sigma(u(s), v(s))$. We calculate

$$\dot{\gamma}(s) = \dot{u}(s)\,\sigma_u + \dot{v}(s)\,\sigma_v, \tag{13}$$

$$\ddot{\gamma}(s) = \ddot{u}(s)\,\sigma_u + \ddot{v}(s)\,\sigma_v + \dot{u}(s)^2\,\sigma_{uu} + 2\,\dot{u}(s)\,\dot{v}(s)\,\sigma_{uv} + \dot{v}(s)^2\,\sigma_{vv}. \tag{14}$$

Let T, N be the unit tangent and normal of the curve $\gamma(s)$ at $\gamma(s_0) = p_0$, and denote by $N_S := \frac{\sigma_u \times \sigma_v}{\|\sigma_u \times \sigma_v\|}$ the unit normal at $p_0 = \sigma(u_0, v_0)$. As we require $\gamma(s)$ to be tangent to a fixed direction, $\dot{u}(s_0), \dot{v}(s_0)$ are fixed. Therefore we further denote

$$u_1 := \dot{u}(s_0), \qquad v_1 := \dot{v}(s_0) \tag{15}$$

to emphasize this point. Thus we have

$$\ddot{\gamma}(s_0) = \ddot{u}(s_0)\,\sigma_u + \ddot{v}(s_0)\,\sigma_v + u_1^2\,\sigma_{uu} + 2\,u_1\,v_1\,\sigma_{uv} + v_1^2\,\sigma_{vv}. \tag{16}$$

Next observe that $N \parallel \ddot{\gamma}(s_0) \perp T,\ T \perp N_S$. We see that

$$\kappa \geqslant |\ddot{\gamma}(s_0) \cdot N_S| = |\mathbb{L}\,u_1^2 + 2\,\mathbb{M}\,u_1\,v_1 + \mathbb{N}\,v_1^2| \tag{17}$$

thanks to the fact that $\sigma_u \cdot N_S = \sigma_v \cdot N_S = 0$.

As σ_u, σ_v form a basis of $T_{p_0}S$, it is always possible to find $\ddot{u}(s_0), \ddot{v}(s_0)$ such that $\ddot{\gamma}(s_0) \parallel N_S$. Consequently, we conclude (when $\|u_1\,\sigma_u + v_1\,\sigma_v\| = 1$)

$$\kappa_{\min}(u_1\,\sigma_u + v_1\,\sigma_v) = |\mathbb{L}\,u_1^2 + 2\,\mathbb{M}\,u_1\,v_1 + \mathbb{N}\,v_1^2|. \tag{18}$$

Remark 3. An arc-length parametrized curve $\gamma(s) = \sigma(u(s), v(s))$ satisfy $\kappa(s) = |\kappa_{\min}(T(s))|$ at every s if and only if $u(s), v(s)$ satisfy the following equations

$$\frac{\mathrm{d}}{\mathrm{d}s}(\mathbb{E}\,\dot{u} + \mathbb{F}\,\dot{v}) = \frac{1}{2}\,(\mathbb{E}_u\,(\dot{u})^2 + 2\,\mathbb{F}_u\,\dot{u}\,\dot{v} + \mathbb{G}_u\,(\dot{v})^2), \tag{19}$$

$$\frac{\mathrm{d}}{\mathrm{d}s}(\mathbb{F}\,\dot{u} + \mathbb{G}\,\dot{v}) = \frac{1}{2}\,(\mathbb{E}_v\,(\dot{u})^2 + 2\,\mathbb{F}_v\,\dot{u}\,\dot{v} + \mathbb{G}_v\,(\dot{v})^2). \tag{20}$$

Exercise 3. Prove this.

- Normal and geodesic curvatures.

DEFINITION 4. *Let $\gamma(t) := \sigma(u(t), v(t))$ be a curve in S passing $p_0 = \sigma(u(t_0), v(t_0))$. Denote by T, N the unit tangent direction and unit normal direction of $\gamma(t)$ at p_0, and by N_S the unit normal direction of S at p_0. Denote by κ the curvature of $\gamma(t)$ at p_0. Then*

$$\kappa\,N = \kappa_n\,N_S + \kappa_g\,(N_S \times T). \tag{21}$$

We call κ_n the ***normal curvature*** *and κ_g the* ***geodesic curvature*** *of $\gamma(t)$ at p_0.*

- Properties.
  - There holds
  $$\kappa^2 = \kappa_n^2 + \kappa_g^2. \tag{22}$$
  - $|\kappa_n|$ is the smallest possible curvature for all curves in S passing p_0 with $\dot{\gamma}(t)$ parallel to the fixed direction $w_0 \in T_p(S)$.
  - Let $w_0 \in T_{p_0}S$ be fixed. Let $\gamma(t)$ be the intersection of S with the plane passing p_0 spanned by w_0 and N_S[1]. Then the curvature of $\gamma(t)$ at p_0 is $|\kappa_n|$.

    Warning

    The curvature of $\gamma(t)$ at $p \neq p_0$ may not equal to $|\kappa_n(p)|$ anymore.

    Exercise 4. Find an example illustrating this. (One possibility is cylinder).
  - In general, we have
  $$\kappa_n = \kappa \cos\psi, \quad \kappa_g = \pm\kappa \sin\psi \tag{23}$$
  where ψ is the angle between N_S and N.

    In particular, if $\gamma(t)$ is the intersection of S with a plane passing the line through p_0 in the direction w, then the curvature of $\gamma(t)$ at p_0 is given by
  $$\kappa = \frac{|\kappa_n|}{\cos\psi} \tag{24}$$
  where ψ is the angle between the plane and the unit normal N_S to the surface at p_0.

1. Such $\gamma(t)$ is called a "normal section"