Math 348 Differential Geometry of Curves and Surfaces

Lecture 10 Applications of The First Fundamental Form

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Please do not hesitate to interrupt me if you have a question.

Review

Definition and Formulas for the First Fundamental Form

$$\boxed{\mathbb{E} = \sigma_u \cdot \sigma_u, \mathbb{F} = \sigma_u \cdot \sigma_v, \mathbb{G} = \sigma_v \cdot \sigma_v.}$$

- **Intuition.**

  Different "scale" at different points on the map.

- **Definition.** At each $p \in S$, a bilinear form on T_pS.

  $v, w \in T_pS : v = v_1\sigma_u + v_2\sigma_v, w = w_1\sigma_u + w_2\sigma_v.$

$$\langle v, w\rangle_{p,S} = \mathbb{E}v_1w_1 + \mathbb{F}(v_1w_2 + v_2w_1) + \mathbb{G}v_2w_2$$

- **Measurements.**
  1. Arc length: $L = \int_a^b \langle\dot{\gamma}, \dot{\gamma}\rangle_{p,S}^{1/2}\mathrm{d}t$;
  2. Angle: $\cos\angle(v, w) = \frac{\langle v,w\rangle_{p,S}}{\langle v,v\rangle_{p,S}^{1/2}\langle w,w\rangle_{p,S}^{1/2}}$;
  3. Area: $A = \int_U \sqrt{\mathbb{E}\mathbb{G} - \mathbb{F}^2}\mathrm{d}u\mathrm{d}v$.

Properties of The First Fundamental Form

Change of variables

$\boxed{\sigma : U \mapsto S \text{ with } \mathbb{E}, \mathbb{F}, \mathbb{G}; \text{ Change of variables: } (u, v) \to (\tilde{u}, \tilde{v}). \quad \tilde{\mathbb{E}}, \tilde{\mathbb{F}}, \tilde{\mathbb{G}}?}$

- **Intuition.**

  How to put "scale" onto a second map?
- **Formulas.**

  $u = U(\tilde{u}, \tilde{v}), v = V(\tilde{u}, \tilde{v})$. Then substitute $\mathrm{d}u = \frac{\partial U}{\partial \tilde{u}}\mathrm{d}\tilde{u} + \frac{\partial U}{\partial \tilde{v}}\mathrm{d}\tilde{v}$ and $\mathrm{d}v = \frac{\partial V}{\partial \tilde{u}}\mathrm{d}\tilde{u} + \frac{\partial V}{\partial \tilde{v}}\mathrm{d}\tilde{v}$ into $\mathbb{E}\mathrm{d}u^2 + 2\mathbb{F}\mathrm{d}u\mathrm{d}v + \mathbb{G}\mathrm{d}v^2$ to obtain $\tilde{\mathbb{E}}, \tilde{\mathbb{F}}, \tilde{\mathbb{G}}$.

Example

$\mathrm{d}u^2 + \mathrm{d}v^2$ under polar coordinates $u = r\cos\theta, v = r\sin\theta$ can be obtained as follows.

$$\mathrm{d}u = \cos\theta\mathrm{d}r - r\sin\theta\mathrm{d}\theta, \qquad \mathrm{d}v = \sin\theta\mathrm{d}r + r\cos\theta\mathrm{d}\theta.$$

This gives

$$\mathrm{d}u^2 + \mathrm{d}v^2 = \mathrm{d}r^2 + r^2\mathrm{d}\theta^2.$$

Isometry

$\boxed{f : S_1 \mapsto S_2\text{: Arc length of } \gamma \text{ equals that of } f(\gamma) \text{ for any } \gamma.}$

- **How to check.**
  $f : S_1 \mapsto S_2$ is an isometry $\Leftrightarrow$ their first fundamental forms coincide, in the sense that for any surface patch σ_1 for S_1, if we let $\sigma_2 = f \circ \sigma_1$, then $\mathbb{E}_1 = \mathbb{E}_2$, $\mathbb{F}_1 = \mathbb{F}_2$, $\mathbb{G}_1 = \mathbb{G}_2$.

Example

Let S_1 be the plane region $\{(x, y, z) \mid z = 0, y \in (-\pi, \pi)\}$, let S_2 be cylinder $x^2 + y^2 = 1$. Then $f(x, y, z) = (\cos y, \sin y, x)$ is an isometry between them.

Conformal mapping

$\boxed{f: S_1 \mapsto S_2\text{: Angle between } \gamma, \tilde{\gamma} \text{ preserved.}}$

- **How to check.**
  $f : S_1 \mapsto S_2$ is conformal $\Leftrightarrow$ their first fundamental forms are proportional, that is for any surface patch σ_1 for S_1, if we let $\sigma_2 = f \circ \sigma_1$, then there is a function $\lambda(u, v)$ such that
  $$\mathbb{E}_2 = \lambda \mathbb{E}_1, \mathbb{F}_2 = \lambda \mathbb{F}_1, \mathbb{G}_2 = \lambda \mathbb{G}_1.$$

Example

$f(x, y, 0) = \left(\frac{2x}{1+x^2+y^2}, \frac{2y}{1+x^2+y^2}, \frac{x^2+y^2-1}{1+x^2+y^2}\right)$ is conformal between the plane $z = 0$ and the unit sphere.

Theorem

Let S_1, S_2 be arbitrary surfaces. Then there is a conformal mapping between them (locally).

Proof.

See lecture notes. $\square$

Equiareal mapping

$\boxed{f : S_1 \mapsto S_2\text{: Area of } \Omega \subseteq S_1 \text{ equals the area of } f(\Omega).}$

- **How to check.**
  $f : S_1 \mapsto S_2$ is equiareal $\Leftrightarrow$ for any surface patch σ_1 of S_1, if we let $\sigma_2 = f \circ \sigma_1$, then
  $$\mathbb{E}_1\mathbb{G}_1 - \mathbb{F}_1^2 = \mathbb{E}_2\mathbb{G}_2 - \mathbb{F}_2^2.$$

Example

S_1: The part $-1 < z < 1$ of the cylinder with the unit circle as its base; S_2: unit sphere. Consider $f(x, y, z) = (\sqrt{1-z^2}x, \sqrt{1-z^2}y, z)$. Then simple calculation shows that f is equiareal.

Existence of equiareal mapping between a plane region and a surface patch $\sigma \Leftrightarrow$ Existence of vector functions U, V such that the first fundamental form for $\tilde{\sigma}(u, v) := \sigma(U(u, v), V(u, v))$ satisfies $\tilde{\mathbb{E}}\tilde{\mathbb{G}} - \tilde{\mathbb{F}}^2 = 1$. This gives the following partial differential equation $\det J^2 = \frac{1}{\mathbb{E}\mathbb{G}-\mathbb{F}^2}$, where J is the Jacobian matrix.

Relations between isometric, conformal, equiareal mappings

$$\boxed{f : S_1 \mapsto S_2.}$$

- **Isometry $\Rightarrow$ Conformal and Equiareal.**
- **Conformal $+$ Equiareal $\Rightarrow$ Isometric.**

Developable Surfaces

Cylinder, Cone, Tangent developable

- **Generalized cylinder.** $\sigma(u, v) = \gamma(u) + v\mathbf{a},\ \|a\| = 1.$
  The first fundamental form is $\mathrm{d}u^2 + \mathrm{d}v^2$. We see that it is isometric to the plane.
- **Generalized cone.** $\sigma(u, v) = v\gamma(u), \|\gamma\| = 1.$
  The first fundamental form is $v^2\mathrm{d}u^2 + dv^2$. We see that it is isometric to the plane.
- **Tangent developable.** $\sigma(u, v) = \gamma(u) + v\dot{\gamma}(u).$
  - u is arc length for γ.
  - $\mathbb{E} = 1 + v^2\kappa^2$, $\mathbb{F} = 1$, $\mathbb{G} = 1$.
  - Isometric to the plane.

No other surfaces are developable

Theorem

Any sufficiently small open subset of a surface locally isometric to a plane is an open subset of a plane, a generalized cylinder, a generalized cone, or a tangent developable.

Proof.

1. S must be a "ruled surface": $\sigma(u, v) = \gamma(u) + vl(u)$[1]. Can assume $\|l\| = 1$.
2. A ruled surface is developable $\Leftrightarrow$ $N(u_0, v)$ is independent of v.
3. A ruled surface is developable $\Leftrightarrow$ $(\dot\gamma \times l) \cdot \dot l = 0$.
4. $\dot\gamma \cdot (l \times \dot l) = 0$. Two cases.
   - 4.1 $\dot l \times l = 0$. Then $l(u)$ is constant. Generalized cylinder.
   - 4.2 $\dot\gamma = a(u)l + b(u)\dot l$. Let $\beta(u) = \gamma(u) - b(u)l(u)$. Calculate $\dot\beta = (a - \dot b)l$. Thus either $l \parallel \dot\beta$, which gives tangent developable, or $\dot\beta = 0$, which gives generalized cone.

□

[1]Will be proved in later lectures

Looking Back and Forward

Summary

$\boxed{\text{Required: §6.1; Optional: §6.2 – 6.5}}$

- **Definitions.**
  - Isometry: Preserves arc length.

$$\mathbb{E}_1 = \mathbb{E}_2, \qquad \mathbb{F}_1 = \mathbb{F}_2, \qquad \mathbb{G}_1 = \mathbb{G}_2.$$

  - Conformal: Preserves angle.

$$\mathbb{E}_1 = \lambda \mathbb{E}_2, \qquad \mathbb{F}_1 = \lambda \mathbb{F}_2, \qquad \mathbb{G}_1 = \lambda \mathbb{G}_2.$$

  - Equiareal: Preserves area.

$$\mathbb{E}_1 \mathbb{G}_1 - \mathbb{F}_1^2 = \mathbb{E}_2 \mathbb{G}_2 - \mathbb{F}_2^2.$$

See you next Tuesday!

$\boxed{\text{The second fundamental form } \langle\!\langle \cdot, \cdot \rangle\!\rangle.}$

1. Measuring curving of a surface.
2. Another bilinear form $\mathbb{L}\mathrm{d}u^2 + 2\mathbb{M}\mathrm{d}u\mathrm{d}v + \mathbb{N}dv^2$.