Lecture 10: Examples and Applications of $\langle \cdot, \cdot \rangle_{p,S}$

Disclaimer. As we have a textbook, this lecture note is for guidance and supplement only. It should not be relied on when preparing for exams.

In this lecture we apply the first fundamental form to understand surface patch.

The required textbook sections are §6.1. The optional sections are §6.2–§6.5.

I try my best to make the examples in this note different from examples in the textbook. Please read the textbook carefully and try your hands on the exercises. During this please don't hesitate to contact me if you have any questions.

Table of contents

Formulas for the First Fundamental Form

- Calculate the first fundamental form: Let $\sigma: U \mapsto \mathbb{R}^3$ be a surface patch of S.

$$\mathbb{E}(u,v)\,\mathrm{d}u^2 + 2\,\mathbb{F}(u,v)\,\mathrm{d}u\,\mathrm{d}v + \mathbb{G}(u,v)\,\mathrm{d}v^2 \tag{1}$$

with

$$\mathbb{E} = \|\sigma_u\|^2, \qquad \mathbb{F} = \sigma_u \cdot \sigma_v, \qquad \mathbb{G} = \|\sigma_v\|^2. \tag{2}$$

- Use the first fundamental form to calculate length, angle, area.
  - Arc length for the curve $\gamma(t) := \sigma(u(t), v(t))$ from $t = a$ to $t = b$.

$$L = \int_a^b \sqrt{\mathbb{E}(\gamma(t))\,\dot{u}(t)^2 + 2\,\mathbb{F}(\gamma(t))\,\dot{u}(t)\,\dot{v}(t) + \mathbb{G}(\gamma(t))\,\dot{v}(t)^2}\,\mathrm{d}t. \tag{3}$$

  - Angle between $\gamma_1(t) := \sigma(u_1(t), v_1(t))$ and $\gamma_2(t) := \sigma(u_2(t), v_2(t))$. Assume the two curves intersect at $p = \sigma(u_0, v_0) = \gamma_1(t_1) = \gamma_2(t_2)$.

$$\cos\theta = \frac{\mathbb{E}\,\dot{u}_1(t_1)\,\dot{u}_2(t_2) + \mathbb{F}\,(\dot{u}_1(t_1)\,\dot{v}_2(t_2) + \dot{u}_2(t_1)\,\dot{v}_1(t_2)) + \mathbb{G}\dot{v}_1(t_1)\,\dot{v}_2(t_2)}{\sqrt{\mathbb{E}\,\dot{u}_1(t_1)^2 + 2\,\mathbb{F}\,\dot{u}_1(t_1)\,\dot{v}_1(t_1) + \mathbb{G}\,\dot{v}_1(t_1)^2}\,\sqrt{\mathbb{E}\,\dot{u}_2(t_2)^2 + 2\,\mathbb{F}\,\dot{u}_2(t_2)\,\dot{v}_2(t_2) + \mathbb{G}\,\dot{v}_2(t_2)^2}} \tag{4}$$

  Here $\mathbb{E} = \mathbb{E}(u_0, v_0)$, $\mathbb{F} = \mathbb{F}(u_0, v_0)$, $\mathbb{G} = \mathbb{G}(u_0, v_0)$.

  - Area of $\sigma(U)$.

$$\int_U \sqrt{\mathbb{E}(\sigma(u,v))\,\mathbb{G}(\sigma(u,v)) - \mathbb{F}(\sigma(u,v))^2}\,\mathrm{d}u\,\mathrm{d}v. \tag{5}$$

1. First fundamental form under re-parametrization

- A common situation is the following. Say we have the first fundament form of a surface patch $\sigma(u,v)$ as $\mathbb{E}(u,v), \mathbb{F}(u,v), \mathbb{G}(u,v)$. Now the surface is re-parametrized through $u = U(\tilde{u}.\tilde{v})$ and $v = V(\tilde{u}, \tilde{v})$. How to obtain the new first fundamental form?
- The simplest way is to do the following: Substitute $u = U(\tilde{u}.\tilde{v})$, $v = V(\tilde{u}, \tilde{v})$, and

$$\mathrm{d}u = U_{\tilde{u}}\,\mathrm{d}\tilde{u} + U_{\tilde{v}}\,\mathrm{d}\tilde{v}, \qquad \mathrm{d}v = V_{\tilde{u}}\,\mathrm{d}\tilde{u} + V_{\tilde{v}}\,\mathrm{d}\tilde{v} \tag{6}$$

  into the formula

$$\mathbb{E}\,\mathrm{d}u^2 + 2\,\mathbb{F}\,\mathrm{d}u\,\mathrm{d}v + \mathbb{G}\,\mathrm{d}v^2 \tag{7}$$

  to obtain

$$\tilde{\mathbb{E}}\,\mathrm{d}\tilde{u}^2 + 2\,\tilde{\mathbb{F}}\,\mathrm{d}\tilde{u}\,\mathrm{d}\tilde{v} + \tilde{\mathbb{G}}\,\mathrm{d}\tilde{v}^2. \tag{8}$$

1. Use first fundamental form to understand surfaces (optional).

1.1. Isometry

DEFINITION 1. (DEFINITION 6.2.1 OF TEXTBOOK) *$f: S_1 \mapsto S_2$ is called a local isometry if it takes any curve in S_1 to a curve of the same length in S_2.*

Notice that

- If $f\colon S_1 \mapsto S_2$ is a local isometry, then f is one-to-one, that is if $p, q \in S_1$ are different points, then so are $f(p), f(q) \in S_2$.
- As a consequence, if $\sigma_1(u, v)$ is a surface patch for S_1, then $\sigma_2(u, v) := f(\sigma_1(u, v))$ is a surface patch for S_2.
- Now let $\mathbb{E}_1\, \mathrm{d}u^2 + 2\,\mathbb{F}_1\, \mathrm{d}u\, \mathrm{d}v + \mathbb{G}_1\, \mathrm{d}v^2$ be the first fundamental form of S_1 calculated using σ_1, and $\mathbb{E}_2\, \mathrm{d}u^2 + 2\,\mathbb{F}_2\, \mathrm{d}u\, \mathrm{d}v + \mathbb{G}_2\, \mathrm{d}v^2$ be the first fundamental form of S_2 calculated using σ_2.
- For any curve $\sigma_1(u(t), v(t))$ on S_1, it is mapped to $\sigma_2(u(t), v(t))$ on S_2. For any $a < b$, the arc length of the two curves are given by

$$\int_a^b \sqrt{\mathbb{E}_1\, \dot{u}^2 + 2\,\mathbb{F}_1 \dot{u}\, \dot{v} + \mathbb{G}_1\, \dot{v}^2}\, \mathrm{d}t \tag{9}$$

  and

$$\int_a^b \sqrt{\mathbb{E}_2\, \dot{u}^2 + 2\,\mathbb{F}_2\, \dot{u}\, \dot{v} + \mathbb{G}_2\, \dot{v}^2}\, \mathrm{d}t \tag{10}$$

  respectively.
- As they are equal for any $a < b$, there must hold

$$\mathbb{E}_1\, \dot{u}^2 + 2\,\mathbb{F}_1\, \dot{u}\, \dot{v} + \mathbb{G}_1\, \dot{v}^2 = \mathbb{E}_2\, \dot{u}^2 + 2\,\mathbb{F}_2\, \dot{u}\, \dot{v} + \mathbb{G}_2\, \dot{v}^2 \tag{11}$$

  for every t.
- Now we fix $(u_0, v_0) = (u(t_0), v(t_0))$ and note that (11) must hold for every $(u(t), v(t))$ passing through this point, and furthermore $\mathbb{E}_1, \mathbb{F}_1, \mathbb{G}_1, \mathbb{E}_2, \mathbb{F}_2, \mathbb{G}_2$ depend on (u_0, v_0) only.
  - Take $u(t) = u_0 + (t - t_0), v(t) = v_0$. We obtain $\mathbb{E}_1 = \mathbb{E}_2$.
  - Take $u(t) = u_0$, $v(t) = v_0 + (t - t_0)$. We obtain $\mathbb{G}_1 = \mathbb{G}_2$.
  - Take $u(t) = u_0 + t - t_0$, $v(t) = v_0 + t - t_0$. We obtain $\mathbb{F}_1 = \mathbb{F}_2$.

  This means the following must hold.

$$\mathbb{E}_1 = \mathbb{E}_2, \qquad \mathbb{F}_1 = \mathbb{F}_2, \qquad \mathbb{G}_1 = \mathbb{G}_2. \tag{12}$$

- Obviously, if (12) holds, then f is a local isometry.

Summarizing the above, we see that

THEOREM 2. (COROLLARY 6.2.3 OF TEXTBOOK) *Let $f\colon S_1 \mapsto S_2$ be a local diffeomorphism. It is a local isometry if and only if for every surface patch σ_1 of S_1, the patches σ_1 and $\sigma_2 := f \circ \sigma_1$ of S_1 and S_2 respectively, have the same first fundamental form.*

PROPOSITION 3. *Let $f\colon S_1 \mapsto S_2$ be a local diffeomorphism, then*

i. it preserves angles. That is if $x_1(t), x_2(t)$ are two intersecting curves on S_1, then the angle between them at the intersection is the same as the angle between $f(x_1(t))$ and $f(x_2(t))$ on S_2.

ii. it preserves area. That is the area of of a region Ω on S_1 is the same as that of $f(\Omega)$ on S_2.

Exercise 1. Prove Proposition 3.

Example 4. There is a local isometry between the cylinder $x_1^2 + x_2^2 = 1$ and the plane $x_3 = 0$. To see this, we take S_1 to be the cylinder and S_2 the plane. We use the following surface patches

$$\sigma_1(u, v) = (\cos u, \sin u, v), \qquad \sigma_2(u, v) = f(\sigma_1(u, v)) = (u, v, 0). \tag{13}$$

Exercise 2. What is f in the original x_1, x_2, x_3 variables?

Now calculate

$$\mathbb{E}_1 = 1, \mathbb{F}_1 = 0, \mathbb{G}_1 = 1; \qquad \mathbb{E}_2 = 1, \mathbb{F}_2 = 0, \mathbb{G}_2 = 1. \tag{14}$$

Remark 5. We notice that there is a issue here. In Theorem 2 we require $\mathbb{E}_1 = \mathbb{E}_2$, etc., for every σ_1. Could it happen that for one σ_1 we have $\mathbb{E}_1 = \mathbb{E}_2, \ldots$ at $p \in S_1$ but for another $\tilde{\sigma}_1$ of S_1 covering p this ceases to hold? We check that such cannot happen. Let's say $\tilde{\sigma}_1$ and σ_1 are related through $u = U(\tilde{u}, \tilde{v})$ and $v = V(\tilde{u}, \tilde{v})$. Then we have

$$\mathrm{d}u = U_{\tilde{u}}\, \mathrm{d}\tilde{u} + U_{\tilde{v}}\, \mathrm{d}\tilde{v}, \qquad \mathrm{d}v = V_{\tilde{u}}\, \mathrm{d}\tilde{u} + V_{\tilde{v}}\, \mathrm{d}\tilde{v}. \tag{15}$$

This leads to

$$\tilde{\mathbb{E}}_1 = \mathbb{E}_1\, U_{\tilde{u}}^2 + 2\, \mathbb{F}_1\, U_{\tilde{u}}\, V_{\tilde{u}} + \mathbb{G}_1\, V_{\tilde{u}}^2, \qquad \tilde{\mathbb{E}}_2 = \mathbb{E}_2\, U_{\tilde{u}}^2 + 2\, \mathbb{F}_2\, U_{\tilde{u}}\, V_{\tilde{u}} + \mathbb{G}_2\, V_{\tilde{u}}^2. \tag{16}$$

Thus we see that $\tilde{\mathbb{E}}_1 = \tilde{\mathbb{E}}_2$. The arguments for $\tilde{\mathbb{F}}_1, \tilde{\mathbb{F}}_2, \ldots$ are similar.

1.2. Conformal mappings

Definition 6. (Definition 6.3.2 of the textbook) *$f: S_1 \mapsto S_2$ is conformal if the angle of intersection at p for $\gamma_1, \tilde{\gamma}_1$ is always the same as the angle at $f(p)$ of $f(\gamma_1), f(\tilde{\gamma}_1)$.*

Theorem 7. (Corollary 6.3.4 of the textbook) *A local diffeomorphism $f: S_1 \mapsto S_2$ is conformal if and only if for every surface patch σ_1 of S_1, the first fundamental forms of the patches σ_1 and $\sigma_2 := f \circ \sigma_1$ are proportional. In other words, there is a function $\lambda(u, v)$ such that*

$$\mathbb{E}_2 = \lambda\, \mathbb{E}_1, \qquad \mathbb{F}_2 = \lambda\, \mathbb{F}_1, \qquad \mathbb{G}_2 = \lambda\, \mathbb{G}_1. \tag{17}$$

Proof. We have, for all $(v_1, v_2), (w_1, w_2)$,

$$\frac{\mathbb{E}_1 v_1\, w_1 + \mathbb{F}_1(v_1\, w_2 + v_2\, w_1) + \mathbb{G}_1\, v_2\, w_2}{\sqrt{\mathbb{E}_1\, v_1^2 + 2\, \mathbb{F}_1\, v_1\, v_2 + \mathbb{G}_1\, v_2^2}\, \sqrt{\mathbb{E}_1\, w_1^2 + 2\, \mathbb{F}_1\, w_1\, w_2 + \mathbb{G}_1\, w_2^2}} \tag{18}$$

equals

$$\frac{\mathbb{E}_2\, v_1\, w_1 + \mathbb{F}_2(v_1\, w_2 + v_2\, w_1) + \mathbb{G}_2\, v_2\, w_2}{\sqrt{\mathbb{E}_2\, v_1^2 + 2\, \mathbb{F}_2\, v_1\, v_2 + \mathbb{G}_2\, v_2^2}\, \sqrt{\mathbb{E}_2\, w_1^2 + 2\, \mathbb{F}_2\, w_1\, w_2 + \mathbb{G}_2\, w_2^2}}. \tag{19}$$

Let $v_1 = 1, v_2 = 0$. The above reduces to

$$\frac{\mathbb{E}_1 w_1 + \mathbb{F}_1 w_2}{\sqrt{\mathbb{E}_1} \sqrt{\mathbb{E}_1 w_1^2 + 2 \mathbb{F}_1 w_1 w_2 + \mathbb{G}_1 w_2^2}} = \frac{\mathbb{E}_2 w_1 + \mathbb{F}_2 w_2}{\sqrt{\mathbb{E}_2} \sqrt{\mathbb{E}_2 w_1^2 + 2 \mathbb{F}_2 w_1 w_2 + \mathbb{G}_2 w_2^2}} \tag{20}$$

which further reduces to

$$\mathbb{E}_2 (\mathbb{E}_1 w_1 + \mathbb{F}_1 w_2)^2 (\mathbb{E}_2 w_1^2 + 2 \mathbb{F}_2 w_1 w_2 + \mathbb{G}_2 w_2^2) = \mathbb{E}_1 (\mathbb{E}_2 w_1 + \mathbb{F}_2 w_2)^2 (\mathbb{E}_1 w_1^2 + 2 \mathbb{F}_1 w_1 w_2 + \mathbb{G}_1 w_2^2). \tag{21}$$

Now we compare the terms.

- w_1^4: $\mathbb{E}_1^2 \mathbb{E}_2^2 = \mathbb{E}_2^2 \mathbb{E}_1^2$;
- $w_1^3 w_2$: $\mathbb{E}_1^2 \mathbb{E}_2 \mathbb{F}_2 + \mathbb{E}_2^2 \mathbb{E}_1 \mathbb{F}_1 = \mathbb{E}_2^2 \mathbb{E}_1 \mathbb{F}_1 + \mathbb{E}_1^2 \mathbb{E}_2 \mathbb{F}_2$;
- $w_1^2 w_2^2$: $\mathbb{E}_1^2 \mathbb{E}_2 \mathbb{G}_2 + 2 \mathbb{E}_1 \mathbb{E}_2 \mathbb{F}_1 \mathbb{F}_2 = \mathbb{E}_2^2 \mathbb{E}_1 \mathbb{G}_1 + 2 \mathbb{E}_2 \mathbb{E}_1 \mathbb{F}_2 \mathbb{F}_1 \Longrightarrow \mathbb{E}_1 \mathbb{G}_2 = \mathbb{E}_2 \mathbb{G}_1$ that is $\mathbb{E}_2 / \mathbb{E}_1 = \mathbb{G}_2 / \mathbb{G}_1$; Let the ratio be λ.
- w_2^4: $\mathbb{E}_2 \mathbb{F}_1^2 \mathbb{G}_2 = \mathbb{E}_1 \mathbb{F}_2^2 \mathbb{G}_1 \Longrightarrow \mathbb{F}_2 / \mathbb{F}_1 = \pm\lambda$.

Finally, setting $w_1 = 0, w_2 = 1$ we see that $\mathbb{F}_2 / \mathbb{F}_1 = \lambda$. □

Exercise 3. Address the issue raised in Remark 5 for conformal mappings.

Example 8. There is a conformal mapping between the sphere and the plane. Let S_1 be the plane $x_3 = 0$ and S_2 be the unit sphere $x_1^2 + x_2^2 + x_3^2 = 1$. Define

$$f(x_1, x_2, 0) = \left(\frac{2 x_1}{1 + x_1^2 + x_2^2}, \frac{2 x_2}{1 + x_1^2 + x_2^2}, \frac{x_1^2 + x_2^2 - 1}{x_1^2 + x_2^2 + 1} \right). \tag{22}$$

We take

$$\sigma_1(u, v) = (u, v, 0). \tag{23}$$

Thus

$$\sigma_2(u, v) = \left(\frac{2u}{u^2 + v^2 + 1}, \frac{2v}{u^2 + v^2 + 1}, \frac{u^2 + v^2 - 1}{u^2 + v^2 + 1} \right). \tag{24}$$

We have

$$\mathbb{E}_2 = \frac{4}{(u^2 + v^2 + 1)^2}, \tag{25}$$

$$\mathbb{F}_2 = 0, \tag{26}$$

$$\mathbb{G}_2 = \frac{4}{(u^2 + v^2 + 1)^2}. \tag{27}$$

On the other hand clelary $\mathbb{E}_1 = \mathbb{G}_1 = 1, \mathbb{F}_1 = 0$. Therefore the mapping f is conformal.

Exercise 4. Show that the usual "spherical coordinate" is not a conformal mapping.

Theorem 9. *Let S_1, S_2 be arbitrary surfaces. Then there is a conformal mapping between them (locally).*

Proof. We sketch the proof.

1. It suffices to prove the theorem for the case S_1 is the plane $x_3 = 0$. We identify this plane with the plane of the parameters u, v.
2. We first show that for any surface S and any $p \in S$, there exists a surface patch σ such that $\sigma_u \perp \sigma_v$ everywhere.
3. We prove the following: Let $a(u,v), b(u,v) \in T_{\sigma(u,v)} S$, not parallel. Then there is a re-parametrization $\tilde{\sigma}(\tilde{u}, \tilde{v}) := \sigma(U(\tilde{u}, \tilde{v}), V(\tilde{u}, \tilde{v})$ around p such that $\tilde{\sigma}_{\tilde{u}} \parallel a, \tilde{\sigma}_{\tilde{v}} \parallel b$.

   To see this, we check that all we need is for U, V to satisfy

$$\frac{\partial(U,V)}{\partial(\tilde{u},\tilde{v})} = \begin{pmatrix} \lambda\, a_1 & \lambda\, a_2 \\ \mu\, b_1 & \mu\, b_2 \end{pmatrix} \tag{28}$$

   or equivalently, there exist integration factors λ, μ such that $\frac{(b_2, -b_1)}{\lambda\,(a_1\, b_2 - a_2\, b_1)} = \nabla \tilde{U}$ and $\frac{(-a_2, a_1)}{\mu(a_1\, b_2 - a_2\, b_1)} = \nabla \tilde{V}$ where $\big(\tilde{U}, \tilde{V}\big)$ is the inverse of the function $(\tilde{u}, \tilde{v}) \mapsto (U, V)$.
4. Recall that $(f_1(u,v), f_2(u,v))$ is the gradient of a function if and only if $\frac{\partial f_1}{\partial v} = \frac{\partial f_2}{\partial u}$ which reduces to a first order linear partial differential equation for λ (or μ). The existence of solution for such an equation is guaranteed.
5. We have reduced the first fundamental form to $\mathbb{E}\, \mathrm{d}u^2 + \mathbb{G}\, \mathrm{d}v^2$. Now let $\lambda^2 := \frac{\mathbb{G}}{\mathbb{E}}$. We have $\mathbb{E}\, (\mathrm{d}u - i\, \lambda\, \mathrm{d}v)\, (\mathrm{d}u + i\, \lambda\, \mathrm{d}v)$. Similar to above we have a (complex) change of variable such that the first fundamental form becomes $\tilde{\mathbb{E}}\, \mathrm{d}\tilde{u}\, \mathrm{d}\tilde{v}$ where there holds $\tilde{v} = \overline{\tilde{u}}$ the conjugate of $\tilde{u}$. Now setting the new variables $\bar{u}, \bar{v}$ to be $\tilde{u} = \bar{u} + i\, \bar{v}$ gives the desired result. □

Example 10. (Mercator projection[1]) Let S_1 be the unit sphere and S_2 be the cylinder $x_1^2 + x_2^2 = 1$. Let $p \in S_1$ and let θ be the angle from the x_1-x_2 plane to the ray connecting the origin to p. Then $f: (x_1, x_2, x_3) \mapsto \left(\frac{x_1}{\sqrt{x_1^2 + x_2^2}}, \frac{x_2}{\sqrt{x_1^2 + x_2^2}}, \ln\big(\tan\big(\frac{\theta}{2} + \frac{\pi}{4}\big)\big) \right)$ is conformal.

1.3. Equiareal mappings

Definition 11. (Definition 6.4.4 of the textbook) *$f: S_1 \mapsto S_2$ is said to be equiareal if it takes any region in S_1 to a region of the same area in S_2.*

Theorem 12. (Theorem 6.4.5) *A local diffeomorphism $f: S_1 \mapsto S_2$ is equiareal if and only if, for any surface patch σ_1 of S_1, the first fundamental forms of σ_1 and $\sigma_2 = f \circ \sigma_1$ satisfy*

$$\mathbb{E}_1\, \mathbb{G}_1 - \mathbb{F}_1^2 = \mathbb{E}_2\, \mathbb{G}_2 - \mathbb{F}_2^2 \tag{29}$$

Exercise 5. Address the issue raised in Remark 5 for equiareal mappings.

Remark 13.

- $f: S_1 \mapsto S_2$ is a local isometry if and only if it is conformal and equiareal.

1. http://www.math.ubc.ca/~israel/m103/mercator/mercator.html.

- There are equiareal mappings between the sphere and the plane.[2]

QUESTION 14. *Are there always equiareal mapping between two surfaces?* [3]

2. Developable surfaces (optional)

In this section we try to understand which surfaces can be "flattened" without stretching or squeezing. In other words, which surface has a local isometry with the plane. Such a surface is called "developable". Let S be a developable surface. Then we have the following.

- S is a "ruled" surface, that is S can be covered by surface patches of the form

$$\sigma(u,v)=\gamma(u)+v\,l(u) \tag{31}$$

  where $\gamma(u)$ is a curve in $\mathbb{R}^3$ and $l(u)$ is a curve on $\mathbb{S}^2$. The proof of this involves Gaussian curvature and may be discussed in a few weeks.[4]

- A ruled surface S: $\sigma(u,v)=\gamma(u)+v\,l(u)$ is developable if and only if $N(\sigma(u_0,v))$ is independent of v, that is the tangent planes along the straightline does not rotate. To see this, observe the following.

  - The image of each line $\gamma(u_0)+v\,l(u_0)$ is also a straightline in the plane. Assume otherwise, let $\sigma(u_0,v_1)$ and $\sigma(u_0,v_2)$ be such that the straight line connecting them is mapped to a curve not straight. Then the pre-image of the straightline connecting $f(\sigma(u_0,v_1))$ and $f(\sigma(u_0,v_2))$ is shorter than $|v_1-v_2|$ but this is not possible as $|v_1-v_2|$ is the shortest distance between $\sigma(u_0,v_1)$ and $\sigma(u_0,v_2)$ in $\mathbb{R}^3$ (not just on S).

  - Let $v_1\neq v_2$. Then clearly $l(u_0)\in T_{\sigma(u_0,v_1)}S$ and also $l(u_0)\in T_{\sigma(u_0,v_2)}S$. Now start from $\sigma(u_0,\,v_1)$ draw a curve (to one side of l) perpendicular to $l(u_0)$, then start from $\sigma(u_0,v_2)$ draw on the same side a curve perpendicular to $l(u_0)$. As isometries are conformal, the images of these two curves on the plane are also perpendicular to the image of l (which is a straight line). Considering the distance between two points on the two curves very close to the line will show that the tangents of the two curves must be parallel and consequently the two tangent planes coincide.

- A ruled surface S: $\sigma(u,v)=\gamma(u)+v\,l(u)$ is developable if and only if $(\dot{\gamma}(u)\times l(u))\cdot\dot{l}(u)=0$.

  Proof. We calculate

$$\sigma_u=\dot{\gamma}(u)+v\,\dot{l}(u), \qquad \sigma_v=l(u). \tag{32}$$

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2. https://en.wikipedia.org/wiki/Lambert_azimuthal_equal-area_projection. Also see Theorem 6.4.6 of the textbook.

3. The existence of such a mapping between S and a plane region is equivalent to finding $U(u,v),V(u,v)$ such that $\tilde{\sigma}(u,v):=\sigma(U(u,v),V(u,v))$ gives $\tilde{\mathbb{E}}\,\tilde{\mathbb{G}}-\tilde{\mathbb{F}}^2=1$. This reduces to the equation

$$\det\begin{pmatrix} U_u & U_v \\ V_u & V_v \end{pmatrix}=\pm\frac{1}{\sqrt{\mathbb{E}\,\mathbb{G}-\mathbb{F}^2}}. \tag{30}$$

4. Also see http://web.mit.edu/hyperbook/Patrikalakis-Maekawa-Cho/node190.html .

Thus

$$\sigma_u \times \sigma_v = \dot{\gamma}(u) \times l(u) + v\,\dot{l}(u) \times l(u). \tag{33}$$

We calculate

$$(\sigma_u \times \sigma_v)_v = \dot{l}(u) \times l(u). \tag{34}$$

Notice that $0 = (\dot{\gamma}(u) \times l(u)) \cdot \dot{l}(u) = -\dot{\gamma}(u) \cdot \big(\dot{l}(u) \times l(u)\big)$. Consequently $\dot{\gamma}(u) \times l(u) \parallel \dot{l}(u) \times l(u)$. From this it follows that $[\sigma_u \times \sigma_v] \times [\sigma_u \times \sigma_v]_v = 0$ which implies that the direction of $\sigma_u \times \sigma_v$ does not change as v changes. □

- We remark that the only ruled surfaces that allow two (or more) different ways of ruling it are the hyperboloid of a single sheet, the hyperbolic paraboloid, and the plane. The former two are not developable, while the last is obviously developable.
- We have the following.

Theorem 15. *Any sufficiently small open subset of a surface locally isometric to a plane is an open subset of a plane, a generalized cylinder, a generalized cone, or a tangent developable.*

Proof. All we need to show is if S is a ruled surface and is developable, then S is one of the following:

- plane;
- generalized cylinder: $\gamma(u) + v\,l_0$;
- generalized cone: $\gamma_0 + v\,l(u)$;
- tangent developable: $\gamma(u) + v\,\dot{\gamma}(u)$.

To see this we discuss the possible cases for $(\dot{\gamma}(u) \times l(u)) \cdot \dot{l}(u) = \dot{\gamma}(u) \cdot \big(\dot{l}(u) \times l(u)\big) = 0$.

- $\dot{l} \times l = 0$. In this case $l(u)$ is constant and we have generalized cylinder;
- $\dot{l} \times l \neq 0$. Now as $\dot{\gamma} \perp \dot{l} \times l$, we have $\dot{\gamma}(u) = a(u)\; l(u) + b(u)\; \dot{l}(u)$. Now let $\beta(u) := \gamma(u) - b(u)\,l(u)$. We calculate

$$\dot{\beta}(u) = \big(a(u) - \dot{b}(u)\big)\,l(u). \tag{35}$$

  - If $a(u) - \dot{b}(u) = 0$, then $\beta(u)$ is a fixed point and we have generalized cone.
  - If $a(u) - \dot{b}(u) \neq 0$, then $l(u) \parallel \dot{\beta}(u)$ and the surface becomes

$$\beta(u) + [b(u) + v]\,l(u) \tag{36}$$

    which is the same surface as

$$\beta(u) + v\,\dot{\beta}(u) \tag{37}$$

    a tangent developable. □