Math 348 Differential Geometry of Curves and Surfaces

Lecture 9 The First Fundamental Form

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Table of contents

Please do not hesitate to interrupt me if you have a question.

Review of Midterm 1

Review of Midterm 1

High: 15 (100%); Low: 3 (20%); Average: 11.6 (77%); Median: 12 (80%)

1. Problem 1. Average: 4.8; Median: 5;
2. Problem 2. Average: 3.8; Median: 4;
3. Problem 3. Average: 3.1; Median: 3;
4. Problem 4. Average: 0; Median: 0.

Problem 2

$$\boxed{\sigma(u,v) = (u, v, u^2 - v^2);\ D_p\mathcal{G}(w),\ p = (2,1,3), w = (1,2,0).}$$

1. Find $\sigma(u_0, v_0) = p$; $u_0 = 2, v_0 = 1$.
2. Calculate $\sigma_u(u_0, v_0), \sigma_v(u_0, v_0)$; $(1, 0, 4), (0, 1, -2)$.
3. Find a, b, $w = a\sigma_u(u_0, v_0) + b\sigma_v(u_0, v_0)$; $a = 1, b = 2$.
4. Calculate $N(u,v) = \frac{\sigma_u \times \sigma_v}{\|\sigma_u \times \sigma_v\|} = \frac{(-2u, 2v, 1)}{\sqrt{1+4u^2+4v^2}}$. ;
5. Calculate $N_u(u_0, v_0) = 21^{-3/2}(-10, -16, -8), N_v(u_0, v_0) = 21^{-3/2}(16, 34, -4)$.;
6. $$D_p\mathcal{G}(a\sigma_u + b\sigma_v) = aN_u + bN_v = 21^{-3/2}(22, 52, -16).$$

Problem 2: Alternative solution

$$\boxed{\sigma(u,v) = (u, v, u^2 - v^2);\ D_p\mathcal{G}(w),\ p = (2,1,3), w = (1,2,0).}$$

1. Find u_0, v_0 such that $\sigma(u_0, v_0) = p$: $u_0 = 2, v_0 = 1$.
2. Calculate $\sigma_u(u_0, v_0) = (1, 0, 4), \sigma_v(u_0, v_0) = (0, 1, -2)$;
3. Find a, b such that $w = a\sigma_u(u_0, v_0) + b\sigma_v(u_0, v_0)$: $a = 1, b = 2$.
4. Calculate $DF(u_0, v_0)$ for $F = (\tilde{\sigma})^{-1} \circ \mathcal{G} \circ \sigma$;
   4.1 $\mathcal{G}(p) = 21^{-1/2}(-4, 2, 1) \Rightarrow \tilde{\sigma}(\tilde{u}, \tilde{v}) = (\tilde{u}, \tilde{v}, \sqrt{1 - \tilde{u}^2 - \tilde{v}^2})$;
   4.2 $F(u, v) = (\tilde{\sigma})^{-1}(N(u, v)) = (1 + 4u^2 + 4v^2)^{-1/2}(-2u, 2v)$;
   4.3 $DF(u_0, v_0) = 21^{-3/2}\begin{pmatrix} -10 & 16 \\ -16 & 34 \end{pmatrix}$
5. Calculate $\begin{pmatrix} \tilde{a} \\ \tilde{b} \end{pmatrix} = DF(u_0, v_0) \cdot \begin{pmatrix} a \\ b \end{pmatrix} = 21^{-3/2}\begin{pmatrix} 22 \\ 52 \end{pmatrix}$
6. Find $\tilde{u}_0, \tilde{v}_0$ such that $\tilde{\sigma}(\tilde{u}_0, \tilde{v}_0) = \mathcal{G}(p)$: $\tilde{u}_0 = -4/\sqrt{21}, \tilde{v}_0 = 2/\sqrt{21}$.
7. Calculate $\tilde{\sigma}_u(\tilde{u}_0, \tilde{v}_0) = (1, 0, 4), \tilde{\sigma}_v(\tilde{u}_0, \tilde{v}_0) = (0, 1, -2)$;
8. $$D_pf(w) = \tilde{a}\tilde{\sigma}_u(\tilde{u}_0, \tilde{v}_0) + \tilde{b}\tilde{\sigma}_v(\tilde{u}_0, \tilde{v}_0) = 21^{-3/2}(22, 52, -16).$$

Problem 3

$$\boxed{\gamma(t) = (a\cos t, a\sin t, bt) \text{ has } \kappa(t) = 1, \tau(t) = -2.}$$

- Note that $\gamma(t)$ is not arc length parametrized.
- Either parametrize by arc length first, or use the correct general formula!

Problem 4

$\kappa(t) = 2\tau(t)$. Find constant vector forming fixed angle with $B(t)$.

- Need w_0 be a constant vector!
- γ can be reparametrized by arc length.
- **Guess.** $\frac{\mathrm{d}}{\mathrm{d}s}(T + 2B) = 0$.
- **More deductive.**
  1. $\frac{\mathrm{d}}{\mathrm{d}s}(w_0 \cdot B) = 0$;
  2. $\Rightarrow w_0 \cdot N = 0$;
  3. $\Rightarrow -\kappa(w_0 \cdot T) + \tau(w_0 \cdot B) = 0$;
  4. $\Rightarrow 2(w_0 \cdot T) = w_0 \cdot B$;
  5. $\Rightarrow w_0 \parallel T + 2B$;
  6. Check $\dot{w}_0 = 0$.

The First Fundamental Form

Measurement on Surfaces

$$\boxed{S : \sigma(u, v).}$$

Distance, Angle, Area?

- **Analogy.** Scale on a map;
- **The First Fundamental Form.**
  - Three "scale" factors $\mathbb{E}, \mathbb{F}, \mathbb{G}$ at every point on the map;
  - Bilinear form on T_pS: $\mathbb{E}(u, v)\mathrm{d}u^2 + 2\mathbb{F}(u, v)\mathrm{d}u\mathrm{d}v + \mathbb{G}(u, v)\mathrm{d}v^2$.
  - Alternative notation.

$$\langle v, w\rangle_{p,S}.$$

  - For all measurements, only need to know $\mathbb{E}(u, v), \mathbb{F}(u, v), \mathbb{G}(u, v)$. Does not need to know $\sigma(u, v)$.

Arc Length

- $\gamma(t) = \sigma(u(t), v(t))$.
- Arc length.

$$\begin{aligned} L &= \int_a^b \sqrt{\dot{\gamma} \cdot \dot{\gamma}} \mathrm{d}t && (1) \\ &= \int_a^b \sqrt{\|\sigma_u\|^2 \dot{u}^2 + 2\sigma_u \cdot \sigma_v \dot{u}\dot{v} + \|\sigma_v\|^2 \dot{v}^2} \mathrm{d}t. && (2) \end{aligned}$$

- Define

$$\mathbb{E}(u, v) = \|\sigma_u\|^2, \qquad \mathbb{F}(u, v) = \sigma_u \cdot \sigma_v, \qquad \mathbb{G}(u, v) = \|\sigma_v\|^2.$$

then

$$L = \int_a^b \sqrt{\mathbb{E}\dot{u}^2 + 2\mathbb{F}\dot{u}\dot{v} + \mathbb{G}\dot{v}^2} \mathrm{d}t$$

Angle and Area

$$v, w \in T_pS.\ \cos\angle(v,w)?$$

- $v = v_1\sigma_u + v_2\sigma_v, w = w_1\sigma_u + w_2\sigma_v$;
- $v \cdot w = \|\sigma_u\|^2 v_1w_1 + \sigma_u \cdot \sigma_v(v_1w_2 + v_2w_1) + \|\sigma_v\|^2 v_2w_2$;
- Thus

$$\cos\angle(v,w) = \frac{\mathbb{E}v_1w_1 + \mathbb{F}(v_1w_2 + v_2w_1) + \mathbb{G}v_2w_2}{\sqrt{\mathbb{E}v_1^2 + 2\mathbb{F}v_1v_2 + \mathbb{G}v_2^2}\sqrt{\mathbb{E}w_1^2 + 2\mathbb{F}w_1w_2 + \mathbb{G}w_2^2}}$$

$$\text{Area of } \sigma(U)?$$

-

$$A = \int_U \|\sigma_u \times \sigma_v\| \, \mathrm{d}u\mathrm{d}v \quad (3)$$

$$= \int_U \sqrt{\mathbb{E}\mathbb{G} - \mathbb{F}^2}\mathrm{d}u\mathrm{d}v. \quad (4)$$

The First Fundamental Form

$$\boxed{v = v_1\sigma_u + v_2\sigma_v, w = w_1\sigma_u + w_2\sigma_v.}$$

- **Definition.**

$$\langle v, w\rangle_{p,S} = \mathbb{E}v_1w_1 + \mathbb{F}(v_1w_2 + v_2w_1) + \mathbb{G}v_2w_2$$

- $\langle v, w\rangle_{p,S} = v \cdot w$;
- **Measurements.**
  - Arc length: $L = \int_a^b \langle\dot\gamma, \dot\gamma\rangle_{p,S}^{1/2}\mathrm{d}t$;
  - Angle: $\cos\angle(v, w) = \frac{\langle v,w\rangle_{p,S}}{\langle v,v\rangle_{p,S}^{1/2}\langle w,w\rangle_{p,S}^{1/2}}$;
  - Area: $A = \int_U \sqrt{\mathbb{E}\mathbb{G} - \mathbb{F}^2}\mathrm{d}u\mathrm{d}v$.
- **Significance.**
  - $\mathbb{E}, \mathbb{F}, \mathbb{G}$ only depend on the surface.
  - (Will see) $\mathbb{E}, \mathbb{F}, \mathbb{G}$ contains information on how the surface is curved.

Examples

Example

1. Plane. $\sigma(u, v) = (u, v, 1 + 2u + 3v)$;
2. Cylinder. $\sigma(u, v) = (\cos u, \sin u, v)$;
3. Sphere. $\sigma(u, v) = (u, v, \sqrt{1 - u^2 - v^2})$;
4. Torus. $\sigma(u, v) = ((a + b\cos u)\cos v, (a + b\cos u)\sin v, b\sin u)$;
5. $\sigma(u, v) = \left(\frac{2u}{1+u^2+v^2}, \frac{2v}{1+u^2+v^2}, \frac{u^2+v^2-1}{1+u^2+v^2}\right)$.

$$\boxed{\mathbb{E} = 4u^2, \mathbb{F} = uv, \mathbb{G} = v^2}$$

- $\gamma : \sigma(u(t), v(t))$, $u(t) = v(t) = t$. Arc length from $t = 1$ to $t = 2$.

$$\int_0^1 \sqrt{4u(t)^2\dot{u}^2 + 2u(t)v(t)\dot{u}\dot{v} + v(t)^2\dot{v}^2}\mathrm{d}t = 3\sqrt{7}/2;$$

- Angle between $\gamma_1 : u_1(t) = t, v_1(t) = 1$ and $\gamma_2 : u_2(t) = 1, v_2(t) = t$.
  1. Intersection: $u = 1, v = 1$;
  2. At intersection $\mathbb{E} = 4, \mathbb{F} = \mathbb{G} = 1$;
  3. At intersection $\dot{u}_1 = 1, \dot{v}_1 = 0$, $\dot{u}_2 = 0, \dot{v}_2 = 1$.
  4. cos of the angle is

$$\frac{4 \times 1 \times 0 + 1 \times (1 \times 1 + 0 \times 0) + 1 \times 0 \times 1}{\sqrt{4 \times 1 \times 1 + 1 \times 1 \times 0 + 1 \times 0 \times 0}\sqrt{4 \times 0 \times 0 + 1 \times 0 \times 1 + 1 \times 1 \times 1}} = \frac{1}{2}.$$

- Area of $\sigma(U)$ where $U = [0, 1] \times [0, 1]$.

$$\int_U \sqrt{\mathbb{E}\mathbb{G} - \mathbb{F}^2} = \sqrt{3}/4.$$

Looking Back and Forward

Required: §6.1; Optional: §6.2 – 6.5

- **Definitions.**
  - The first fundamental form.

$$\langle v, w\rangle_{p,S} = \mathbb{E}v_1w_1 + \mathbb{F}(v_1w_2 + v_2w_1) + \mathbb{G}v_2w_2$$

- **Formulas.**
-

$$\mathbb{E}(u,v) = \|\sigma_u\|^2, \qquad \mathbb{F}(u,v) = \sigma_u \cdot \sigma_v, \qquad \mathbb{G}(u,v) = \|\sigma_v\|^2.$$

- Arc length: $L = \int_a^b \langle\dot\gamma, \dot\gamma\rangle_{p,S}^{1/2}\mathrm{d}t$;
- Angle: $\cos\angle(v,w) = \frac{\langle v,w\rangle_{p,S}}{\langle v,v\rangle_{p,S}^{1/2}\langle w,w\rangle_{p,S}^{1/2}}$;
- Area: $A = \int_U \sqrt{\mathbb{E}\mathbb{G} - \mathbb{F}^2}\mathrm{d}u\mathrm{d}v$.

See you this Thursday!

Understanding and Applying $\langle \cdot, \cdot \rangle_{p,S}$.

1. Re-parametriztion;
2. Isometries;
3. Conformal mappings;
4. Equiareal mappings;
5. Developable surfaces.