Math 348 Differential Geometry of Curves and Surfaces

Lecture 8 Curve Theory II. Frenet-Serret Equations and Applications

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Sept. 28, 2017

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Please do not hesitate to interrupt me if you have a question.

Background Review

$\gamma(s)$: Curve with arc length parametriztion

- Tangent: $T(s) = \dot{\gamma}(s)$;
- Normal: $N(s) = \frac{\ddot{\gamma}(s)}{\|\ddot{\gamma}(s)\|}$;
- Binormal: $B(s) = T(s) \times N(s)$;
- Curvature: Measures "curving", $\kappa(s) = \|\ddot{\gamma}(s)\|$;
- Torsion: Measures "twisting", $\tau(s) = \frac{(\dot{\gamma}(s) \times \ddot{\gamma}(s)) \cdot \dddot{\gamma}(s)}{\kappa(s)^2} = B(s) \cdot \frac{\dddot{\gamma}(s)}{\kappa(s)}$.

Example

$\gamma(s) = (a \cos t, a \sin t, bt)$.

Understanding Curves via Curvature and Torsion

Zero Curvature $\Rightarrow$ Straight

Example

A curve with zero curvature is (part of) a straight line.

Proof.

1. Let the curve by parametrized by arc length: $\gamma(s)$;
2. Zero curvature $\Rightarrow \ddot{\gamma}(s) = 0$;
3. $\dot{\gamma}(s)$ is a constant vector, denote it by T_0;
4. $\gamma(s) = \gamma_0 + \int_0^t T_0 = \gamma_0 + T_0 t$.

$\square$

Zero Torsion $\Rightarrow$ Planar

Example

A curve with zero torsion and nonzero curvature is a plane curve.

Proof.

1. Recall that $\dot{B}(s) = -\tau(s)N(s)$;
2. $B(s) = B_0$ constant vector;
3. $\dot{\gamma}(s) \perp B_0$;
4. Fix s_0. $\forall s, (\gamma(s) - \gamma(s_0)) \perp B_0$.

□

What if ”nonzero curvature” is dropped?

Circle

Example

A curve with zero torsion and nonzero constant curvature is part of a circle.

Proof.

1. We know that it is a plane curve;
2. Let $O(s) := \gamma(s) + \kappa^{-1}N(s)$.
3. $\dot{N}(s) = \frac{\mathrm{d}}{\mathrm{ds}}(B(s) \times T(s))$.
4. Calculate $\dot{O}(s) \cdot N(s) = 0$.
5. Calculate $\dot{O}(s) \cdot T(s) = 0$.

□

Frenet-Serret System

Frenet-Serret System

$\gamma(s)$: Arc length parametrized.

$$\begin{aligned} \dot{T}(s) &= \kappa(s)N(s), & (1) \\ \dot{N}(s) &= -\kappa(s)T(s) + \tau(s)B(s), & (2) \\ \dot{B}(s) &= -\tau(s)N(s). & (3) \end{aligned}$$

- **Analogy.** A car on the road carrying an orthonormal frame.

An Example

Example

Let $\gamma(s)$ be arc length parametrized. Prove that γ is a spherical curve if and only if

$$\frac{\tau}{\kappa} = \frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\dot{\kappa}}{\tau\kappa^2}\right).$$

Proof.

Let $\rho := 1/\kappa, \sigma := 1/\tau$.

- **Only if.**
  1. $\|\gamma - \gamma_0\| = r \Rightarrow T \cdot (\gamma - \gamma_0) = 0, N \cdot (\gamma - \gamma_0) = -\rho, B \cdot (\gamma - \gamma_0) = -\sigma\dot{\rho}$.
  2. $\rho^2 + (\dot{\rho}\sigma)^2 = r^2$. Differentiate $\Rightarrow$ conclusion.
- **If.**
  1. $\Rightarrow \rho^2 + (\dot{\rho}\sigma)^2 = r^2$;
  2. Let $\gamma_0 = \gamma + \rho N + \sigma\dot{\rho}B$, calculate $\dot{\gamma}_0 = 0$.

□

Other Topics

Fundamental Theorem for Curves

Theorem

- **(Existence)** *Let $\kappa(s)$, $\tau(s)$ be two smooth functions. Further assume $\kappa(s) > 0$. Then there is a curve $\gamma(s)$ with s as its arc length parameter, $\kappa(s)$ as its curvature, and $\tau(s)$ as its torsion.*
- **(Uniqueness)** *Let $\gamma(s)$ and $\tilde{\gamma}(s)$ be two curves parametrized by arc length. If $\kappa(s) = \tilde{\kappa}(s)$ and $\tau(s) = \tilde{\tau}(s)$ for all s, then there is a rigid motion M such that $\tilde{\gamma}(s) = M\gamma(s)$.*

How to find this rigid motion?

Local Canonical Form

- Not possible: Different curves with identical curvature and torsion everywhere;
- Possible: Different curves with identical curvature and torsion at one point;
- Consider $\gamma(s)$ with $\gamma(0) = 0$.
  - Assume $T(0) = (1, 0, 0), N(0) = (0, 1, 0), B(0) = (0, 0, 1)$;
  - Let $\kappa_0 = \kappa(0), \tau_0 = \tau(0)$;
  - We have

$$x(s) = s - \frac{1}{6}\kappa_0^2 s^3 + o(s^3), \quad (4)$$

$$y(s) = \frac{1}{2}\kappa_0 s^2 + \frac{1}{6}\kappa_0' s^3 + o(s^3), \quad (5)$$

$$z(s) = \frac{1}{6}\kappa_0 \tau_0 s^3 + o(s^3). \quad (6)$$

  - The curvature and torsion of $(s, \frac{1}{2}\kappa_0 s^2, \frac{1}{6}\kappa_0\tau_0 s^3)$?

- **Signed curvature.** N_S: T rotates $\pi/2$ counter-clockwise; $\ddot{\gamma} = \kappa_S N_S$;
- **Turning angle.**

$$\kappa_S = \dot{\varphi}.$$

- **Baby Gauss-Bonnet.** Let γ be a simple closed plane curve.

$$\int_\gamma \kappa_S(s)\mathrm{d}s = 2k\pi.$$

- **Other properties.** $\dot{N}_S = -\kappa_S T$.

Looking Back and Forward

Summary

Required Sections: §2.2.

- **Definitions.**
  1. Signed curvature.

     N_S: T rotates $\pi/2$ counter-clockwise; $\ddot{\gamma} = \kappa_S N_S$.
- **Formulas.**
  1. Frenet-Serret System:

$$\begin{aligned} \dot{T}(s) &= \kappa(s)N(s), && (7) \\ \dot{N}(s) &= -\kappa(s)T(s) + \tau(s)B(s), && (8) \\ \dot{B}(s) &= -\tau(s)N(s). && (9) \end{aligned}$$

  2. $\dot{N}_S = -\kappa_S T$.

See You Next Tuesday!

Review for Midterm I

1. Curve theory;
2. Surfaces in Calculus.