Math 348 Fall 2017

Lecture 8: Differential Geometry of Curves II

Disclaimer. As we have a textbook, this lecture note is for guidance and supplement only. It should not be relied on when preparing for exams.

In this lecture we see how curvature and torsion can help us understand curves.

The required textbook section is §2.2.

The examples in this note are mostly different from examples in the textbook. Please read the textbook carefully and try your hands on the exercises. During this please don't hesitate to contact me if you have any questions.

Table of contents

1. Understanding curves via curvature and torsion

Example 1. A curve with zero curvature is (part of) a straight line.

Proof. Let $\gamma(s)$ be arc length parametrization. Then $\|\ddot{\gamma}(s)\| = \kappa(s) = 0 \Longrightarrow \ddot{\gamma}(s) = 0$. Consequently

$$\dot{\gamma}(s) = T_0 \tag{1}$$

a constant vector. Finally by Fundamental Theorem of Calculus,

$$\gamma(s) = \gamma(s_0) + \int_{s_0}^{s} \dot{\gamma}(t)\,\mathrm{d}t = \gamma(s_0) + (s - s_0)\,T_0, \tag{2}$$

which is a straight line. □

Example 2. A curve with zero torsion and nonzero curvature is a plane curve.

Proof. Recall that $\dot{B}(s) = -\tau(s)\;N(s)$, so $B(s) = B_0$, a constant vector. By definition $\dot{\gamma}(s) = T(s) \cdot B_0 = 0$. Now by FTC,

$$(\gamma(s) - \gamma(s_0)) \cdot B_0 = \left[\int_{s_0}^{s} \dot{\gamma}(t)\,\mathrm{d}t\right] \cdot B_0 = \int_{s_0}^{s} [\dot{\gamma}(t) \cdot B_0]\,\mathrm{d}t = 0. \tag{3}$$

Thus $\gamma(s)$ lies in the plane

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot B_0 = \gamma(s_0) \cdot B_0 \tag{4}$$

and the conclusion follows. □

Example 3. A curve with zero torsion and nonzero constant curvature is part of a circle.

Proof. As $\tau = 0$ it is a plane curve, and the binormal $B(s) = B_0$ a constant vector. Let $\gamma_0(s) := \gamma(s) + \kappa^{-1}\,N(s)$. We have

$$\dot{\gamma}_0(s) = T(s) + \kappa^{-1}\dot{N}(s). \tag{5}$$

Now differentiating $N(s) = B_0 \times T$ we obtain

$$\kappa^{-1}\,\dot{N}(s) = \kappa^{-1}\left(B_0 \times \dot{T}\right) = B_0 \times N(s) = -T(s), \tag{6}$$

consequently $\dot{\gamma}_0(s) = 0 \Longrightarrow \gamma_0(s) = \gamma_0$ is a constant vector. Finally we have

$$\|\gamma(s) - \gamma_0\| = \kappa^{-1} \tag{7}$$

a constant, so $\gamma(s)$ is (part of) a circle. □

2. Frenet-Serret equations

One set of ODE determining the evolution of the vectors $T(s), N(s), B(s)$.

- We have seen that the evolution of T, B are governed by

$$\dot{T}(s) = \kappa(s)\,N(s), \qquad \dot{B}(s) = -\tau(s)\,N(s). \tag{8}$$

- For the equation for $N(s)$, we notice that

$$\begin{aligned}\dot{N}(s) &= \frac{\mathrm{d}}{\mathrm{d}t}(B(s)\times T(s)) \\ &= \dot{B}(s)\times T(s)+B(s)\times\dot{T}(s) \\ &= -\tau(s)\,N(s)\times T(s)+B(s)\times[\kappa(s)\,N(s)] \\ &= \tau(s)\,B(s)-\kappa(s)\,T(s). \qquad (9)\end{aligned}$$

Frenet-Serret equations for arc length parametrization.

$$\begin{aligned}T' &= \kappa\,N \\ N' &= -\kappa\,T+\tau\,B\,. \\ B' &= -\tau\,N\end{aligned} \qquad (10)$$

Remark 4. It turns out that (10) completely determines the curve. More specifically we have

THEOREM 5. (THEOREM 2.3.6 OF TEXTBOOK) *Let* $\mathrm{K}(s)>0$ *and* $\mathrm{T}(s)$ *be given for all* $s\in(\alpha,\beta)$. *Let* x_0, T_0, N_0,s_0 *be given where* $x_0\in\mathbb{R}^3$, $T_0\perp N_0$ *are unit vectors, and* $s_0\in(\alpha,\beta)$. *Then there is a unique curve with* s *as its arc length parameter, satisfying* $x(s_0)=x_0$, *with* T_0, N_0 *as its unit tangent and normal vectors at* x_0, *and takes* $\mathrm{K}(s),\mathrm{T}(s)$ *as its curvature and torsion for* $s\in(\alpha,\beta)$.

Exercise 1. Can we drop the "with s as its arc length parameter" part?

Exercise 2. Does the Frenet-Serret equations

$$\begin{aligned}T' &= \kappa\,N \\ N' &= -\kappa\,T+\tau\,B \\ B' &= -\tau\,N\end{aligned} \qquad (11)$$

still hold for general parametrization?

2.1. Examples

Example 6. Consider the curve $(\cos t,\sin t,2\,t)$. We would like to calculate T,N,B,κ,τ.

- Preparation. We calculate

$$\begin{aligned}x'(t) &= (-\sin t,\cos t,2), &(12)\\ x''(t) &= (-\cos t,-\sin t,0), &(13)\\ x'''(t) &= (\sin t,-\cos t,0), &(14)\\ \|x'(t)\| &= \sqrt{5}, &(15)\\ x'(t)\times x''(t) &= (2\sin t,-2\cos t,1), &(16)\\ \|x'(t)\times x''(t)\| &= \sqrt{5}. &(17)\end{aligned}$$

- Thus we have

$$
\begin{aligned}
T(t) &= \frac{x'}{\|x'\|} = \frac{1}{\sqrt{5}}(-\sin t, \cos t, 2), && (18)\\
B(t) &= \frac{x' \times x''}{\|x' \times x''\|} = \frac{1}{\sqrt{5}}(2\sin t, -2\cos t, 1), && (19)\\
N(t) &= B \times T = -(\cos t, \sin t), && (20)\\
\kappa(t) &= \frac{\|x' \times x''\|}{\|x'\|^3} = \frac{1}{5}, && (21)\\
\tau(t) &= \frac{(x' \times x'') \cdot x'''}{\|x' \times x''\|^2} = \frac{2}{5}. && (22)
\end{aligned}
$$

Exercise 3. Consider the curve $x(t) = (\cos t, \sin t, e^t)$. Without doing any calculation, can you predict the behavior of $\kappa(t), \tau(t)$ as $t \longrightarrow \infty$? Check your prediction through calculation.

Example 7. Consider the curve $y = f(x), z = 0$. We parametrize it as $x(t) = (t, f(t), 0)$. Then

$$
\begin{aligned}
x'(t) &= (1, f', 0), && (23)\\
x''(t) &= (0, f'', 0), && (24)\\
x'''(t) &= (0, f''', 0), && (25)\\
\|x'(t)\| &= \sqrt{1 + (f')^2}, && (26)\\
x'(t) \times x''(t) &= (0, 0, f''), && (27)\\
\|x'(t) \times x''(t)\| &= |f''|. && (28)
\end{aligned}
$$

Therefore

$$
\begin{aligned}
T(t) &= \frac{(1, f'(t), 0)}{\sqrt{1 + f'(t)^2}}, && (29)\\
B(t) &= (0, 0, \operatorname{sgn}(f''(t))), && (30)\\
N(t) &= \operatorname{sgn}(f''(t))\,(-f'(t), 1), && (31)\\
\kappa(t) &= \frac{\|x' \times x''\|}{\|x'\|^3} = \frac{|f''|}{\left(\sqrt{1 + (f')^2}\right)^3}, && (32)\\
\tau(t) &= 0. && (33)
\end{aligned}
$$

Example 8. Let $\gamma(s)$ be arc length parametrized. Prove that γ is a spherical curve if and only if

$$\frac{\tau}{\kappa} = \frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\dot{\kappa}}{\tau \kappa^2}\right). \quad (34)$$

Proof.

- Only if.
  We need to prove that $\gamma(s)$ is a spherical curve $\Longrightarrow \frac{\tau}{\kappa} = \frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\dot{\kappa}}{\tau\kappa^2}\right)$.
  As $\gamma(s)$ is spherical, there is a constant vector γ_0 and a constant r such that

$$\|\gamma(s) - \gamma_0\| = r. \quad (35)$$

  Differentiating $(\gamma(s) - \gamma_0) \cdot (\gamma(s) - \gamma_0) = r^2$ we have

$$T(s) \cdot (\gamma(s) - \gamma_0) = 0. \quad (36)$$

Differentiating this we have

$$\kappa(s)\,N(s)\cdot(\gamma(s)-\gamma_0)=-T(s)\cdot T(s)=-1 \tag{37}$$

which gives

$$N(s)\cdot(\gamma(s)-\gamma_0)=-\frac{1}{\kappa(s)}. \tag{38}$$

Differenting (38) we have

$$(-\kappa\,T+\tau\,B)\cdot(\gamma(s)-\gamma_0)=\frac{\dot{\kappa}}{\kappa^2}. \tag{39}$$

Thanks to (36) we reach

$$B\cdot(\gamma(s)-\gamma_0)=\frac{\dot{\kappa}}{\tau\,\kappa^2}. \tag{40}$$

Putting together

$$T(s)\cdot(\gamma-\gamma_0)=0,\;\; N\cdot(\gamma-\gamma_0)=-\frac{1}{\kappa},\;\; B\cdot(\gamma-\gamma_0)=\frac{\dot{\kappa}}{\tau\,\kappa^2} \tag{41}$$

we have

$$\gamma-\gamma_0=-\frac{1}{\kappa}\,N+\frac{\dot{\kappa}}{\tau\,\kappa^2}\,B. \tag{42}$$

Consequently

$$\left(\frac{1}{\kappa}\right)^2+\left(\frac{\dot{\kappa}}{\tau\,\kappa^2}\right)^2=r^2. \tag{43}$$

Taking $\frac{\mathrm{d}}{\mathrm{d}s}$ of this we obtain $\frac{\tau}{\kappa}=\frac{\mathrm{d}}{\mathrm{d}s}\big(\frac{\dot{\kappa}}{\tau\kappa^2}\big)$.

- If.

  We need to prove that $\frac{\tau}{\kappa}=\frac{\mathrm{d}}{\mathrm{d}s}\big(\frac{\dot{\kappa}}{\tau\kappa^2}\big)\Longrightarrow\gamma(s)$ is a spherical curve.

  Thanks to the "only if" part we make the following guesses:

$$\gamma_0(s):=\gamma(s)+\frac{1}{\kappa(s)}\,N(s)-\frac{\dot{\kappa}(s)}{\tau(s)\,\kappa(s)^2}\,B(s), \tag{44}$$

$$r(s):=\sqrt{\left(\frac{1}{\kappa(s)}\right)^2+\left(\frac{\dot{\kappa}(s)}{\tau(s)\,\kappa(s)^2}\right)^2}. \tag{45}$$

It follows that

$$\|\gamma(s)-\gamma_0(s)\|=r(s), \tag{46}$$

and all we need to prove are $\dot{r}(s)=0$, $\dot{\gamma}_0(s)=0$.

i. $\dot{r}(s)=0$. We have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}s}[r(s)^2] &= \frac{\mathrm{d}}{\mathrm{d}s}\left[\left(\frac{1}{\kappa(s)}\right)^2+\left(\frac{\dot{\kappa}(s)}{\tau(s)\,\kappa(s)^2}\right)^2\right]\\ &= -2\,\frac{\dot{\kappa}}{\kappa^2}\,\frac{1}{\kappa}+2\,\frac{\dot{\kappa}}{\tau\kappa^2}\,\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\dot{\kappa}}{\tau\kappa^2}\right)\\ &= \frac{2\,\dot{\kappa}}{\tau\kappa^2}\left[-\frac{\tau}{\kappa}+\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\dot{\kappa}}{\tau\kappa^2}\right)\right]=0.\end{aligned} \tag{47}$$

ii. $\dot{\gamma}_0(s) = 0$. We have

$$\begin{aligned}\dot{\gamma}_0 &= \dot{\gamma} - \frac{\dot{\kappa}}{\kappa^2} N + \frac{1}{\kappa} \dot{N} - \frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\dot{\kappa}}{\tau\kappa^2}\right) B - \frac{\dot{\kappa}}{\tau\kappa^2} \dot{B} \\ &= T - \frac{\dot{\kappa}}{\kappa^2} N + \frac{1}{\kappa}[-\kappa T + \tau B] - \frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\dot{\kappa}}{\tau\kappa^2}\right) B + \frac{\dot{\kappa}}{\kappa^2} N \\ &= 0. \end{aligned} \tag{48}$$

□

3. The local canonical form

- Let $\gamma(s)$ be the arc length parametrization of the curve. We can calculate its Taylor expansion near s_0:

$$\gamma(s) - \gamma(s_0) = \dot{\gamma}(s_0)(s - s_0) + \frac{\ddot{\gamma}(s_0)}{2}(s - s_0)^2 + \frac{\dddot{\gamma}(s_0)}{6}(s - s_0)^3 + R(s, s_0) \tag{49}$$

where $\lim_{s \to s_0} \frac{\|R(s, s_0)\|}{|s - s_0|^3} = 0$.

Now we try to re-write (49) using κ, τ, T, N, B, s, s_0 only. Clearly $\dot{\gamma} = T$ and $\ddot{\gamma} = \kappa N$. Differentiating one more time we have

$$\dddot{\gamma} = \dot{\kappa} N + \kappa \dot{N} = \dot{\kappa} N - \kappa^2 T + \kappa \tau B. \tag{50}$$

Therefore we have

$$\gamma(s) - \gamma(s_0) = a(s, s_0) T(s_0) + b(s, s_0) N(s_0) + c(s, s_0) B(s_0) + R(s, s_0) \tag{51}$$

where

$$a(s, s_0) = (s - s_0) - \frac{\kappa(s_0)^2}{6}(s - s_0)^3, \tag{52}$$

$$b(s, s_0) = \frac{\kappa(s_0)(s - s_0)^2}{2} + \frac{\kappa'(s_0)(s - s_0)^3}{6}, \tag{53}$$

$$c(s, s_0) = \frac{\kappa(s_0)\tau(s_0)(s - s_0)^3}{6}, \tag{54}$$

$$\lim_{s \to s_0} \frac{\|R(s, s_0)\|}{|s - s_0|^3} = 0. \tag{55}$$

4. Plane curves

- For plane curves $B(s)$ is a constant.
- $\{T, N\}$ is not convenient as N may not be counter-clockwise from T.
- Define the "signed normal" N_S to be the vector obtained from T by rotating $\pi/2$ counter-clockwise. Further define the "signed curvature":

$$\dot{T} = \kappa_S N_S. \tag{56}$$

- If we write $T(s) = (\cos\varphi(s), \sin\varphi(s))$, we obtain

$$\dot{T} = (-\dot{\varphi}(s)\sin\varphi(s), \dot{\varphi}(s)\cos\varphi(s)). \tag{57}$$

On the other hand we have

$$N_S = (-\sin\varphi(s), \cos\varphi(s)). \tag{58}$$

Consequently $\kappa_S(s) = \dot{\varphi}(s)$.

- Let γ be a simple closed plane curve. Then $T_{\text{start}} = T_{\text{end}}$ and

$$\int_\gamma \kappa_S(s)\,\mathrm{d}s = 2\,k\,\pi \tag{59}$$

for some $k \in \mathbb{Z}$.

  - We will prove later in the course that $k = 1$;
  - Generalization of (59) to surfaces leads to the Gauss-Bonnet Theorem, which would be the last and biggest theorem of Math 348.