Math 348 Differential Geometry of Curves and Surfaces

Lecture 7 Curve Theory I. Curvature and Torsion

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Please do not hesitate to interrupt me if you have a question.

Background Review

Curves in Calculus

- **Definitions.**
  1. Regular Curve.
     A map $\gamma : (\alpha, \beta) \mapsto \mathbb{R}^n$, for some $-\infty \leqslant \alpha < \beta \leqslant \infty$, such that $\gamma \in C^\infty$, and $\dot{\gamma}(t) \neq 0$ for every $t \in (\alpha, \beta)$.
  2. Arc Length Parametrization.
     $\gamma(s)$ is an arc length parametrization if $\|\dot{\gamma}(s)\| = 1$ at every s.
  3. Tangent Vector.
     $\dot{\gamma}(t)$ is the tangent vector for the curve $\gamma(t)$.
- **Formulas.**
  1. Arc Length.

$$L = \int_a^b \|\dot{\gamma}(t)\| \, \mathrm{d}t.$$

- **Procedures.**
  1. Re-parametrize with arc length.
     1.1 Solve $\dot{S}(t) = \|\dot{\gamma}(t)\|$;
     1.2 Calculate inverse function $T(s)$ of $S(t)$;
     1.3 Re-write $\Gamma(s) = \gamma(T(s))$.

Curvature and the Normal Vector

Curvature for Arc Length Parametrized Curves

$\boxed{\gamma(s)\text{: Arc length parametrized curve.}}$

- **What is curvature?**

  A function describing how $\gamma(s)$ curves.

- **Formula for Curvature.**

$$\kappa(s) := \|\ddot{\gamma}(s)\|\,.$$

- **Why this formula?**

  1. How fast the unit tangent vector turns;
  2. How fast the curve deviates from a straight line.

- **Unit normal.**

$$N(s) = \frac{\ddot{\gamma}(s)}{\kappa(s)}.$$

Examples

$$\boxed{\kappa(s) = \|\ddot{\gamma}(s)\|^{1}.}$$

Example

- Circle;
- $\gamma(t) = (\frac{1}{\sqrt{3}}\cos t + \frac{1}{\sqrt{2}}\sin t, \frac{1}{\sqrt{3}}\cos t, \frac{1}{\sqrt{3}}\cos t - \frac{1}{\sqrt{2}}\sin t)$.
- $\gamma(t) = (e^{t}, e^{-t}, \sqrt{2}t)$.

[1] Only works for arc length parametrized curves!

Curvature for General Curves

$\boxed{\gamma(t)\text{: Not necessarily arc length parametrized.}}$

- The formula.
$$\kappa(t) = \frac{\|\dot{\gamma}(t) \times \ddot{\gamma}(t)\|}{\|\dot{\gamma}(t)\|^3}$$
- Understanding and remembering the formula.
    - What does $\dot{\gamma}(t) \times \ddot{\gamma}(t)$ mean?
    - Why divide by $\|\dot{\gamma}(t)\|^3$?

Torsion and the Binormal Vector

Torsion for Arc Length Parametrized Curves

$\boxed{\gamma(s)\text{: Arc length parametrized curve.}}$

- **What is torsion?**

  A function describing how $\gamma(s)$ twists.

- **The formula.**

$$\tau(s) = \frac{(\dot{\gamma}(s) \times \ddot{\gamma}(s)) \cdot \dddot{\gamma}(s)}{\kappa^2(s)}.$$

- **Why this formula?**
  - Binormal.

$$B(s) := T(s) \times N(s).$$

  - How to measure ”twisting”?
    1. $B'(s) = T(s) \times N'(s)$;
    2. $\tau(s) = -B'(s) \cdot N(s)$.

Examples

$$\boxed{\tau(s) = \frac{(\dot{\gamma}(s) \times \ddot{\gamma}(s)) \cdot \dddot{\gamma}(s)}{\kappa^2(s)}.}$$

Example

- Straight lines;
- Circles;
- Plane curves;
- Helix.

Torsion for General Curves

$\boxed{\gamma(t)\text{: Not necessarily arc length parametrized.}}$

- **The formula.**

$$\tau(t) = \frac{(\dot{\gamma}(t) \times \ddot{\gamma}(t)) \cdot \dddot{\gamma}(t)}{\|\dot{\gamma}(t) \times \ddot{\gamma}(t)\|^2}.$$

- **Understanding and remembering the formula.**
    1. Meaning of $(\dot{\gamma}(t) \times \ddot{\gamma}(t)) \cdot \dddot{\gamma}(t)$: Tendency of $\gamma(t)$ ”leaving” the plane spanned by $\{\dot{\gamma}, \ddot{\gamma}\}$.
    2. Why divide by $\|\dot{\gamma}(t) \times \ddot{\gamma}(t)\|^2$.

$$\tau(t) = \frac{\dot{\gamma} \times \ddot{\gamma}}{\|\dot{\gamma} \times \ddot{\gamma}\|} \cdot \frac{\dddot{\gamma} / \|\dot{\gamma}\|^3}{\kappa(t)}.$$

Extra $\kappa(t)^{-1}$: Factoring out the effect of curvature, want ”pure twisting” rate.[2]

[2] Rotating a candy cane?

Looking Back and Forward

Summary

Required Sections: §2.1, §2.3.

- Tangent, Normal, Binormal.
  - $\gamma(s)$ arc length parametrized:

$$T(s) = \dot{\gamma}(s); \qquad N(s) = \frac{\ddot{\gamma}(s)}{\kappa(s)}; \qquad B(s) = T(s) \times N(s).$$

  - General situation.

$$T(t) = \frac{\dot{\gamma}}{\|\dot{\gamma}\|}, \qquad B(t) = \frac{\dot{\gamma} \times \ddot{\gamma}}{\|\dot{\gamma} \times \ddot{\gamma}\|}, \qquad N(t) = B(t) \times T(t).$$

- Curvature and torsion.
  - $\gamma(s)$ arc length parametrized:

$$\kappa(s) = \|\ddot{\gamma}(s)\|, \qquad \tau(s) = \frac{(\dot{\gamma}(s) \times \ddot{\gamma}(s)) \cdot \dddot{\gamma}(s)}{\kappa^2(s)}.$$

  - General situation.

$$\kappa(t) = \frac{\|\dot{\gamma}(t) \times \ddot{\gamma}(t)\|}{\|\dot{\gamma}(t)\|^3}, \qquad \tau(t) = \frac{(\dot{\gamma}(t) \times \ddot{\gamma}(t)) \cdot \dddot{\gamma}(t)}{\|\dot{\gamma}(t) \times \ddot{\gamma}(t)\|^2}.$$

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See You Next Tuesday!

$\boxed{\text{Differential Geometry of Curves Cont.}}$

1. Understanding curves by curvature and torsion;
2. Frenet-Serret equations;
3. Local canonical form.