Curves I: Curvature and Torsion

Disclaimer. As we have a textbook, this lecture note is for guidance and supplement only. It should not be relied on when preparing for exams.

In this lecture we study how a curve curves. We will show that the curving of a general curve can be characterized by two numbers, the curvature and the torsion.

The required textbook sections are: §2.1, §2.3.

The examples in this note are mostly different from examples in the textbook. Please read the textbook carefully and try your hands on the exercises. During this please don't hesitate to contact me if you have any questions.

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1. Curvature

Curvature measures how quickly a curve turns, or more precisely how quickly the unit tangent vector turns.

1.1. Curvature for arc length parametrized curves

- Consider a curve $\gamma(s)$: $(\alpha, \beta) \mapsto \mathbb{R}^3$. Then the unit tangent vector of $\gamma(s)$ is given by $T(s) := \dot{\gamma}(s)$. Consequently, how quickly $T(s)$ turns can be characterized by the number
$$\kappa(s) := \left\|\dot{T}(s)\right\| = \|\ddot{\gamma}(s)\| \tag{1}$$
which call the curvature of the curve.
- As $\|\dot{\gamma}(s)\| = 1$, we have $\ddot{\gamma}(s) \perp \dot{\gamma}(s)$. This leads to the following definition.

Definition 1. *Let $\gamma(s)$ be a curve parametrized by arc length. Then its curvature is defined as $\kappa(s) := \|\ddot{\gamma}(s)\|$. We further denote by $N(s)$ the unit vector $\ddot{\gamma}(s) / \|\ddot{\gamma}(s)\|$ and call it the normal vector at s. We also denote the unit tangent vector $\dot{\gamma}(s)$ by $T(s)$.*

Formulas for curves in arc length parametrization.

- Curvature.
$$\kappa(s_0) = \|\ddot{\gamma}(s_0)\|. \tag{2}$$
- Tangent and Normal vectors.
$$T(s_0) = \dot{\gamma}(s_0), \qquad N(s_0) = \frac{\ddot{\gamma}(s_0)}{\|\ddot{\gamma}(s_0)\|}. \tag{3}$$

1.2. Alternative characterization of the curvature (optional)

- Consider a curve $\gamma(s)$: $(\alpha, \beta) \mapsto \mathbb{R}^3$. Let $p = \gamma(s_0)$ for some $s_0 \in (\alpha, \beta)$. We try to understand how quickly $\gamma(s)$ turns aways from the tangent line at $x(s_0)$.
- The equation for the tangent line is $\gamma(s_0) + t\,\dot{\gamma}(s_0)$.
- The distance from a point $\gamma(s)$ to the tangent line is
$$d(s) := \|(\gamma(s) - \gamma(s_0)) \times \dot{\gamma}(s_0)\| \tag{4}$$
Note that here we have used the fact that $\dot{\gamma}(s_0)$ is a unit vector.
- Now recall Taylor expansion:
$$\gamma(s) - \gamma(s_0) = \dot{\gamma}(s_0)\,(s - s_0) + \frac{1}{2}\,\ddot{\gamma}(s_0)\,(s - s_0)^2 + R(s, s_0) \tag{5}$$
where $\lim_{s \to s_0} \frac{\|R(s, s_0)\|}{(s - s_0)^2} = 0$.
- Substituting (5) into (4), we see that
$$d(s) = \left\|\frac{(s - s_0)^2}{2}\,(\ddot{\gamma}(s_0) \times \dot{\gamma}(s_0)) + R(s, s_0) \times \dot{\gamma}(s_0)\right\| \tag{6}$$

and consequently

$$\lim_{s\to 0}\frac{d(s)}{(s-s_0)^2/2}=\|\ddot{\gamma}(s_0)\times\dot{\gamma}(s_0)\| \tag{7}$$

Exercise 1. Prove (7).

- Thus we see that the quantity $\|\ddot{\gamma}(s_0)\times\dot{\gamma}(s_0)\|=\kappa(s_0)$ measures how the curve "curves" at the point $\gamma(s_0)$.

Exercise 2. Prove that $\|\ddot{\gamma}(s_0)\times\dot{\gamma}(s_0)\|=\|\ddot{\gamma}(s_0)\|=\kappa(s_0)$.

Exercise 3. (7) can be derived slightly differently as follows.

i. Find T such that $\gamma(s)-[\gamma(s_0)+T\dot{\gamma}(s_0)]\perp\dot{\gamma}(s_0)$.

ii. Then $d(s)=\|\gamma(s)-[\gamma(s_0)+T\dot{\gamma}(s_0)]\|$.

iii. Calculate the limit $\lim\frac{d(s)}{(s-s_0)^2/2}$.

1.3. Examples

Example 2. For the unit circle, the curvature is constantly 1. For a circle with radius R, the curvature is constantly $1/R$.

Example 3. (Shifrin2016) Let $\gamma(t)=\left(\frac{1}{\sqrt{3}}\cos t+\frac{1}{\sqrt{2}}\sin t,\frac{1}{\sqrt{3}}\cos t,\frac{1}{\sqrt{3}}\cos t-\frac{1}{\sqrt{2}}\sin t\right)$. We calculate

- Tangent vector:

$$\dot{\gamma}(t)=\left(-\frac{1}{\sqrt{3}}\sin t+\frac{1}{\sqrt{2}}\cos t,-\frac{1}{\sqrt{3}}\sin t,-\frac{1}{\sqrt{3}}\sin t-\frac{1}{\sqrt{2}}\cos t\right) \tag{8}$$

- $\|\dot{\gamma}(t)\|=1$ so we are already in arc length parametrization.
- We have $\|\ddot{\gamma}(t)\|=1$.

So the curvature of this curve is constantly 1.

Example 4. (Shifrin2016) Let $\gamma(t)=\left(e^t,e^{-t},\sqrt{2}\,t\right)$. We calculate

- Tangent vector:

$$\dot{\gamma}(t)=\left(e^t,-e^{-t},\sqrt{2}\right) \tag{9}$$

  and therefore
- $\|\dot{\gamma}(t)\|=e^t+e^{-t}$.
- Solve

$$S'(t)=e^t+e^{-t} \tag{10}$$

  we obtain $S(t)=e^t-e^{-t}$.
- Solve t as a function of s:

$$e^t=\frac{s+\sqrt{s^2+4}}{2}. \tag{11}$$

- Thus $\gamma(s)$ is given by

$$\left(\frac{s+\sqrt{s^2+4}}{2}, \frac{\sqrt{s^2+4}-s}{2}, \sqrt{2}\ln\left(\frac{s+\sqrt{s^2+4}}{2}\right)\right). \tag{12}$$

- We calculate

$$\dot{\gamma}(s) = \left(\frac{1}{2}+\frac{s}{2\sqrt{s^2+4}}, \frac{s}{2\sqrt{s^2+4}}-\frac{1}{2}, \frac{\sqrt{2}}{\sqrt{s^2+4}}\right). \tag{13}$$

To make sure our calculation is correct, we check

$$\|\dot{\gamma}(s)\| = 1. \tag{14}$$

- Finally we calculate

$$\ddot{\gamma}(s) = \left(8\,(s^2+4)^{-3/2}, 8\,(s^2+4)^{-3/2}, -\sqrt{2}\,s\,(s^2+4)^{-3/2}\right) \tag{15}$$

which gives

$$\kappa(s) = \|\ddot{\gamma}(s)\| = \frac{\sqrt{2}\sqrt{s^2+64}}{\sqrt{s^2+4}^3}. \tag{16}$$

2. Torsion

Torsion measures how quickly a curve "twists".

2.1. The osculating plane

- Motivation. Consider a point on a space curve. We have seen that to measure how quickly it curves, we should measure the rate of change for the unit tangent vector. Similarly, to measure how quickly it "twists", we should measure the change rate of the "tangent plane".

- The osculating plane.

  - Let $\gamma(s)$ be a space curve. Its osculating plane at $\gamma(s_0)$ is the plane passing $\gamma(s_0)$ that is spanned by the unit tangent vector $T(s_0) := \dot{\gamma}(s_0)$ and the unit normal vector $N(s_0) := \frac{\ddot{\gamma}(s_0)}{\|\ddot{\gamma}(s_0)\|}$.

  - We see that the osculating plane contains the tangent line.

  - The unit normal vector of the osculating plane is then given by

$$B(s) := T(s) \times N(s) \tag{17}$$

    which we call the *unit binormal vector* of the curve $\gamma(s)$.

    Exercise 4. Prove that $T(s) = N(s) \times B(s)$, $N(s) = B(s) \times T(s)$.

  - Among all the planes containing the tangent line, the osculating plane is the one that "fits" the curve best. See Exercise 5 below.

2.2. Definition of torsion

- How quickly the osculating plane turns is clearly characterized by how quickly the unit binormal vector turns. We calculate

$$\dot{B}(s) = \dot{T}(s) \times N(s) + T(s) \times \dot{N}(s) = T(s) \times \dot{N}(s). \tag{18}$$

  Now notice that $\|N(s)\| = 1 \Longrightarrow \dot{N}(s) \cdot N(s) = 0$. As $T(s) \cdot N(s) = 0$ too, we see that $\dot{B}(s) \parallel N(s)$. Consequently there is a scalar function $\tau(s)$ such that

$$\dot{B}(s) = -\tau(s)\, N(s). \tag{19}$$

  We call $\tau(s)$ the *torsion* of the curve $\gamma(s)$.[1]

- Formula for $\tau(s)$. We calculate

$$\dot{N}(s) = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\ddot{\gamma}(s)}{\kappa(s)}\right) = \frac{\dddot{\gamma}(s)}{\kappa(s)} - \frac{\dot{\kappa}(s)}{\kappa^2(s)}\, \ddot{\gamma}(s). \tag{20}$$

  Thus

$$\begin{aligned} \tau(s) &= -\big(T(s) \times \dot{N}(s)\big) \cdot N(s) \\ &= -\left(\frac{T(s) \times \dddot{\gamma}(s)}{\kappa(s)} - \frac{\dot{\kappa}(s)}{\kappa^2(s)}\, T(s) \times \ddot{\gamma}(s)\right) \cdot N(s) \\ &= -\frac{\dot{\gamma}(s) \times \dddot{\gamma}(s)}{\kappa(s)} \cdot N(s) \\ &= \frac{(\dot{\gamma}(s) \times \ddot{\gamma}(s)) \cdot \dddot{\gamma}(s)}{\kappa^2(s)}. \end{aligned} \tag{21}$$

Formula for torsion (arc length parametrization).

$$\tau(s) = \frac{(\dot{\gamma}(s) \times \ddot{\gamma}(s)) \cdot \dddot{\gamma}(s)}{\kappa^2(s)} \tag{22}$$

Remark 5. Alternative definition of torsion.

- Distance to osculating plane. We easily see that

$$d(s) = \|(\gamma(s) - \gamma(s_0)) \cdot B(s_0)\| \tag{23}$$

  Exercise 5. Prove that, among all planes passing $\gamma(s_0)$, the osculating plane is the only one satisfying

$$\lim_{s \to s_0} \frac{d(s)}{(s - s_0)^2} = 0. \tag{24}$$

- Taylor expansion of $x(s) - x(s_0)$ to order three:

$$\gamma(s) - \gamma(s_0) = \dot{\gamma}(s_0)\,(s - s_0) + \frac{\ddot{\gamma}(s_0)}{2}\,(s - s_0)^2 + \frac{\dddot{\gamma}(s_0)}{6}\,(s - s_0)^3 + R(s, s_0) \tag{25}$$

  where $\lim_{s \to s_0} \frac{\|R(s, s_0)\|}{(s - s_0)^3} = 0$.

1. There is no particular reason for the negative sign.

- Substituting (25) into (23) we see that

$$\lim_{s \to s_0} \frac{d(s)}{(s-s_0)^3/6} = |\dddot{\gamma}(s_0) \cdot B(s_0)|. \tag{26}$$

Exercise 6. Show that

$$|\dddot{\gamma}(s_0) \cdot B(s_0)| = \left| \frac{(\dot{\gamma}(s) \times \ddot{\gamma}(s)) \cdot \dddot{\gamma}(s)}{\kappa(s)} \right|. \tag{27}$$

Compare to (22). Discuss possible reasons for the difference. Do you think (22) is a more reasonable definition for torsion? Why?

2.3. Examples

Example 6. We calculate the torsion of the curve $\gamma(t) = \left(\frac{1}{\sqrt{3}} \cos t + \frac{1}{\sqrt{2}} \sin t, \frac{1}{\sqrt{3}} \cos t, \frac{1}{\sqrt{3}} \cos t - \frac{1}{\sqrt{2}} \sin t\right)$. Note that we have seen there that t is already the arc length parameter.

$$\dot{\gamma}(t) = \left(-\frac{1}{\sqrt{3}} \sin t + \frac{1}{\sqrt{2}} \cos t, -\frac{1}{\sqrt{3}} \sin t, -\frac{1}{\sqrt{3}} \sin t - \frac{1}{\sqrt{2}} \cos t \right), \tag{28}$$

$$\ddot{\gamma}(t) = \left(-\frac{1}{\sqrt{3}} \cos t - \frac{1}{\sqrt{2}} \sin t, -\frac{1}{\sqrt{3}} \cos t, -\frac{1}{\sqrt{3}} \cos t + \frac{1}{\sqrt{2}} \sin t \right) = -x(t). \tag{29}$$

Thus clearly

$$(\dot{\gamma}(t) \times \ddot{\gamma}(t)) \cdot \dddot{\gamma}(t) = 0 \Longrightarrow \tau(t) = 0. \tag{30}$$

3. Curvature and torsion in general parametrization

3.1. Formulas

- The key idea is that κ, τ, T, N, B should be independent of parametrization. In other words, if $\gamma(t)$ and $\gamma(s)$ are two parametrizations of the same curve, and $p = \gamma(t_0) = \gamma(s_0)$, then we must have $\kappa(t_0) = \kappa(s_0)$, $\tau(t_0) = \tau(s_0)$, and so on.
- We will try to obtain the formulas intuitively here. For rigorous derivation, please see §2.3 of the textbook. In the following let $\gamma(t)$ be a curve not necessarily in arc length parametrization.
  - T is the unit tangent vector. So we must have

$$T(t) = \frac{\dot{\gamma}(t)}{\|\dot{\gamma}(t)\|}. \tag{31}$$

  - B. Since N clearly lies in the plane spanned by $\dot{\gamma}$ and $\ddot{\gamma}$, $B \parallel \dot{\gamma} \times \ddot{\gamma}$. Consequently we must have

$$B(t) = \frac{\dot{\gamma}(t) \times \ddot{\gamma}(t)}{\|\dot{\gamma}(t) \times \ddot{\gamma}(t)\|}. \tag{32}$$

  - N. We have

$$N(t) = B(t) \times T(t) = \frac{(\dot{\gamma}(t) \times \ddot{\gamma}(t)) \times \dot{\gamma}(t)}{\|\dot{\gamma}(t) \times \ddot{\gamma}(t)\| \, \|\dot{\gamma}(t)\|}. \tag{33}$$

- κ. We have $\kappa = \frac{\mathrm{d}T}{\mathrm{d}s} \cdot N = \frac{1}{\|\dot{\gamma}(t)\|} \dot{T} \cdot N$ which gives

$$\kappa(t) = \frac{\ddot{\gamma}(t) \cdot [(\dot{\gamma}(t) \times \ddot{\gamma}(t)) \times \dot{\gamma}(t)]}{\|\dot{\gamma}(t)\|^3 \, \|\dot{\gamma}(t) \times \ddot{\gamma}(t)\|} = \frac{\|\dot{\gamma}(t) \times \ddot{\gamma}(t)\|}{\|\dot{\gamma}(t)\|^3}. \tag{34}$$

- τ. We have $\tau = -\frac{\mathrm{d}B}{\mathrm{d}s} \cdot N$ which gives

$$\begin{aligned} \tau &= -\frac{1}{\|\dot{\gamma}(t)\|^2} \frac{(\dot{\gamma}(t) \times \ddot{\gamma}(t)) \cdot [(\dot{\gamma}(t) \times \ddot{\gamma}(t)) \times \dot{\gamma}(t)]}{\|\dot{\gamma}(t) \times \ddot{\gamma}(t)\|^2} \\ &= -\frac{(\dot{\gamma}(t) \times \dddot{\gamma}(t)) \cdot \ddot{\gamma}(t)}{\|\dot{\gamma}(t) \times \ddot{\gamma}(t)\|^2} \\ &= \frac{(\dot{\gamma}(t) \times \ddot{\gamma}(t)) \cdot \dddot{\gamma}(t)}{\|\dot{\gamma}(t) \times \ddot{\gamma}(t)\|^2}. \end{aligned} \tag{35}$$

Exercise 7. In the above we have used the vector identity

$$(u \times v) \times w = -(v \cdot w)\, u + (u \cdot w)\, v \tag{36}$$

for $u, v, w \in \mathbb{R}^3$. Prove this identity and identify where it is used.

DG of Curves: Formulas for general parametrization

$$\kappa = \frac{\|\dot{\gamma} \times \ddot{\gamma}\|}{\|\dot{\gamma}\|^3}. \tag{37}$$

$$\tau = \frac{(\dot{\gamma} \times \ddot{\gamma}) \cdot \dddot{\gamma}}{\|\dot{\gamma} \times \ddot{\gamma}\|^2}. \tag{38}$$

$$T = \frac{\dot{\gamma}}{\|\dot{\gamma}\|}, \tag{39}$$

$$B = \frac{\dot{\gamma} \times \ddot{\gamma}}{\|\dot{\gamma} \times \ddot{\gamma}\|}, \tag{40}$$

$$N = B \times T. \tag{41}$$

Warning

In exams, the formulas (37) and (38) will be provided, but (38–40) will not be provided.