Math 348 Differential Geometry of Curves and Surfaces

Lecture 6 Isometry

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Sept. 21, 2017

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Please do not hesitate to interrupt me if you have a question.

Brief Review

Functions Between Surfaces

- **Definitions.**
  1. f is smooth $\iff F = (\tilde{\sigma})^{-1} \circ f \circ \sigma$ is smooth.
  2. The Gauss Map $\mathcal{G} : S \mapsto \mathbb{S}^2$, $\mathcal{G}(p) = N(u_0, v_0)$ where $p = \sigma(u_0, v_0)$.
- **Formulas.**
  1. Differential.
     $$D_p f(a\sigma_u + b\sigma_v) = \tilde{a}\tilde{\sigma}_{\tilde{u}} + \tilde{b}\tilde{\sigma}_{\tilde{v}}, \begin{pmatrix} \tilde{a} \\ \tilde{b} \end{pmatrix} = DF(u_0, v_0) \cdot \begin{pmatrix} a \\ b \end{pmatrix}.$$
  2. Unit normal.
     $$N(u, v) = \frac{\sigma_u \times \sigma_v}{\|\sigma_u \times \sigma_v\|}.$$
- **Procedures.**

  Calculate $D_p f$.
  1. Pick σ for S covering p (if not given);
  2. Pick $\tilde{\sigma}$ for $\tilde{S}$ covering $f(p)$ (if not given);
  3. Formulate $F = (\tilde{\sigma})^{-1} \circ f \circ \sigma$;
  4. Calculate $DF(u_0, v_0)$ at $\sigma(u_0, v_0) = p$.

More on the Gauss Map

Properties of the Gauss Map

- Visualization: Catenoid, Cylinder.
- Natural surface patch: $N(u, v)$.
- $T_pS = T_{\mathcal{G}(p)}\mathbb{S}^2$.
- $D_p\mathcal{G}(\sigma_u) = N_u$, $D_p\mathcal{G}(\sigma_v) = N_v$.
- Two ”natural” basis for $T_{f(p)}\mathbb{S}^2$.
    1. $\{N_u, N_v\}$;
    2. $\{\sigma_u, \sigma_v\}$.
- Matrix representation for $D_p\mathcal{G}$ using either basis.

Isometry

Isometry

$S, \tilde{S}$: Surface patches; $f : S \mapsto \tilde{S}$ smooth.

- **Isometry.**
  - f onto;
  - $\gamma(t)$: $\forall$ curve in S. $p = \gamma(t_1), q = \gamma(t_2)$: $\forall$ points on γ.
  - Arc length of γ between p, q equals arc length of $f(\gamma)$ between $f(p), f(q)$.
- **Simple Properties.** f: Isometry.
  - f is bijective.
  - f^{-1} is an isometry between $\tilde{S}$ and S.
- **Implications on Measurements.** f: Isometry between S and $\tilde{S}$.
  - f is area-preserving.
  - f is angle-preserving[1].
- **Isometric Surfaces.**

  S and $\tilde{S}$ isometric: There is an isometry between them.

[1] More precisely, $D_p f$ is angle preserving at every p.

Cartography

Faithful Maps

$$\boxed{\text{A map is a function } f : S_{Map} \mapsto S_{Reality}}$$

- **Ideal Map.**
  Ideally, a map should
  1. be (locally) bijective;
  2. Has the same ”scale” everywhere.

  Existence of an ideal map $\iff$ Existence of a (local) isometry.
- **Our Maps are Not Ideal.**
  - Everyday map distorts distance and area: Thetruesize.
  - Angles are preserved[2].

Can we do better?

[2] Important for navigation!

Mercator Projection.

$$\boxed{\sigma(u, v) = (sechu\ \ \cos v, sechu\ \ \sin v, tanhu).}$$

- **Not As We Thought!** For example, Britannica Kids has it wrong here.
- **Why Not?**

  Consider a general cylindrical projection

$$\sigma(u, v) = (f(u) \cos v, f(u) \sin v, g(u))$$

with $f'(u) > 0$, and $f^2 + g^2 = 1$.

1. Angle preservation: $\|\sigma_u\| = \|\sigma_v\|$.
2. Calculation gives $f'^2 + g'^2 = f^2$.
3. Differentiate $f^2 + g^2 = 1$ to obtain $g' = -ff'/g$.
4. Obtain $f' = f\sqrt{1 - f^2}$.
5. Solve.

No Flat Map for Sphere.

There is no isometry between S and $\tilde{S}$ if S is part of a plane and $\tilde{S}$ is a part of a sphere.

- **Idea of the Proof.**
  1. The shortest curve connecting any two points on a plane is straight;
  2. The shortest curve connecting any two points on a sphere is part of a big circle;
  3. Consider a special ”triangle” to reach contradiction.
- **But We have Proved Nothing!**
  The earth surface is not a sphere – will settle this by the end of the semester.

Geodesics for the Plane

Geodesic: A curve that is the shortest path between two (close enough) points on it.

Example

Geodesics in a plane are straight lines.

Proof.

$$\begin{aligned} L &= \int_a^b \|\dot{\gamma}(t)\| \, \mathrm{d}t \\ &\geqslant \left\| \int_a^b \dot{\gamma}(t) \mathrm{d}t \right\| \\ &= \|\gamma(b) - \gamma(a)\| . \end{aligned}$$

$\square$

Geodesics for the Sphere

1. Can assume the two points lie on the yz plane $(0, y_1, z_1)$, $(0, y_2, z_2)$;
2. $\gamma(t) = (x(t), y(t), z(t), \ t \in [a, b]$ connects the two points: $x(a) = x(b) = 0$, $y(a) = y_1$, $y(b) = y_2$, $z(a) = z_1$, $z(b) = z_2$.
3. $\Gamma(t) = (0, r(t), z(t))$ where $r(t) = \sqrt{x(t)^2 + y(t)^2}$.
4. $\Gamma(t)$ connects the two points, with arc length $\geqslant$ the great arc connecting them;
5. Calculate

$$
\begin{aligned}
L_\Gamma = \int_a^b \left\| \dot{\Gamma}(t) \right\| \mathrm{d}t &= \int_a^b \sqrt{\dot{r}(t)^2 + \dot{z}(t)^2} \mathrm{d}t \\
&= \int_a^b \sqrt{\frac{(x(t)\dot{x}(t) + y(t)\dot{y}(t))^2}{x(t)^2 + y(t)^2} + \dot{z}(t)^2} \mathrm{d}t \\
&\leqslant \int_a^b \sqrt{\dot{x}(t)^2 + \dot{y}(t)^2 + \dot{z}(t)^2} \mathrm{d}t \\
&= \int_a^b \|\dot{\gamma}(t)\| \, \mathrm{d}t = L_\gamma .
\end{aligned}
$$

Looking Back and Forward

Summary

- The Gauss map:

  $\{\sigma_u, \sigma_v\}$ is a basis for both T_pS and $T_{\mathcal{G}(p)}\mathbb{S}^2$;
- Isometry.
  - Definition: $f : S \mapsto \tilde{S}$ conserves arc length;
  - Properties: Area and angels are also conserved;
  - Plane and sphere are not isometric;
  - Will discuss in more details later in the course.

See You Next Tuesday!

$\boxed{\text{Differential Geometry of Curves}}$

1. Curvature;
2. Torsion.