Lecture 6: Isometry

Disclaimer. As we have a textbook, this lecture note is for guidance and supplement only. It should not be relied on when preparing for exams.

In this lecture we finish the preliminary discussion of the Gauss map and introduce the idea of isometry. We also prove that there cannot be a planar map for a spherical region.

The examples in this note are different from examples in the textbook. Please read the textbook carefully and try your hands on the exercises. During this please don't hesitate to contact me if you have any questions.

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1. Properties of the Gauss Map

- Let $\mathcal{G}: S \mapsto \mathbb{S}^2$ be the Gauss map of the surface S. Then by definition $T_p S \perp N(p)$. On the other hand, since $\mathbb{S}^2$ is a sphere, there holds $T_{\mathcal{G}(p)}\mathbb{S}^2 \perp \mathcal{G}(p) = N(p)$. Consequently the two surfaces $T_p S$ and $T_{\mathcal{G}(p)}\mathbb{S}^2$ are parallel, and are in fact the same plane (when viewed as a "velocity space".
- Consequently, we can use the basis for $T_p S$, $\{\sigma_u, \sigma_v\}$ as the basis for $T_{\mathcal{G}(p)}\mathbb{S}^2$. In fact it is benefical to think of $D_p\mathcal{G}$ as a linear transformation from $T_p S$ to itself.
- Let S be parametrized by σ. Then $\mathbb{S}^2$ can be parametrized as $\mathcal{G} \circ \sigma$ which is exactly the function

$$N(u, v) = \frac{\sigma_u \times \sigma_v}{\|\sigma_u \times \sigma_v\|}. \tag{1}$$

  In other words, it is natural to parametrize $\mathbb{S}^2$ by N.
- If we use $\{N_u,\ N_v\}$ as basis for $T_{\mathcal{G}(p)}\mathbb{S}^2$, the matrix representation for $D_p\mathcal{G}$ is the identity matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. To see this, we note that

  i. S is parametrized by σ;

  ii. $\tilde{S} := \mathcal{G}(S) \subseteq \mathbb{S}^2$ is parametrized by $N(u, v)$;

  iii. $N^{-1} \circ \mathcal{G} \circ \sigma = N^{-1} \circ N = I$ the identity mapping.

  iv. Consequently the derivative is also identity.
- If we use $\{\sigma_u,\ \sigma_v\}$ as basis for $T_{\mathcal{G}(p)}\mathbb{S}^2$, things become interesting. Considering the curve $\gamma(t) = \sigma(t, 0)$, we see that the corresponding curve on $\mathbb{S}^2$ is $N(t, 0)$. Consequently we have

$$D_p\mathcal{G}(\sigma_u) = N_u \tag{2}$$

  and similarly

$$D_p\mathcal{G}(\sigma_v) = N_v. \tag{3}$$

  From these we can solve the matrix representation of $D_p\mathcal{G}$. The entries of these matrix are the most important quantities in classical differential geometry. We will discuss much more about this later in the course.

2. Isometry

- $f: S \mapsto \tilde{S}$ is an isometry when it is bijective, and preserves arc length.
- More specifically, for any curve $\gamma(t)$ in S, the arc length of γ from $\gamma(a)$ to $\gamma(b)$ equals the arc length of the curve $f(\gamma(t))$, on $\tilde{S}$, from $f(\gamma(a))$ to $f(\gamma(b))$.
- One can show that if f is an isometry, then f also preserves area, that is, area of $W \subseteq S$ equals the area of $f(W) \subseteq \tilde{S}$ for any W.
- One can also show that if f is an isometry, then at every $p \in S$, $D_p f$ preserves angle. More specifically, if $v_1 = D_p f(u_1)$, $v_2 = D_p f(u_2)$, then $\angle(v_1, v_2) = \angle(u_1, u_2)$.

3. Geodesics on the Plane, Cylinder, and Sphere

3.1. On the plane

- THE PROBLEM.

  Let A, B be two points in $\mathbb{R}^n$. There are infinitely many curves that connect the two points. Find the one with shortest arc length.

- MATHEMATICAL FORMULATION.

  Among all curves $\gamma(t)$ satisfying $\gamma(a) = A,\ \gamma(b) = B$, find the one with minimal

$$L := \int_a^b \|\dot{\gamma}(t)\| \,\mathrm{d}t. \tag{4}$$

- THE SOLUTION.

  We notice that

  i. There holds

$$L \geqslant \left\| \int_a^b \dot{\gamma}(t) \,\mathrm{d}t \right\| = \|\gamma(b) - \gamma(a)\| = \|B - A\|. \tag{5}$$

  ii. For the curve $\gamma_L(t) = \frac{b-t}{b-a}\, y + \frac{t-a}{b-a}\, z$, we have

$$L = \int_a^b \|\dot{\gamma}_L(t)\| \,\mathrm{d}t = \int_a^b \left\| \frac{1}{(b-a)} (z - y) \right\| \mathrm{d}t = \|B - A\|. \tag{6}$$

  Therefore the solution is $\dot{\gamma}_L(t)$ which is a straight line.

  Exercise 1. Prove that $\gamma_L(t)$ is a straight line and find its arc length parametrization.

3.2. On the cylinder

- THE PROBLEM.

  Let A, B be two points in $\mathbb{R}^3$ lying on the cylinder with the base circle centered at the original and with radius 1. Find the shortest path along the cyclinder surface connecting the two points.

- MATHEMATICAL FORMULATION.

  Among all curves $\gamma(t) = (\cos t,\ \sin t,\ z(t))$ satisfying $\gamma(a) = A = (x_A,\ y_A,\ z_A)$, $\gamma(b) = B = (x_B, y_B, z_B)$, find the one with minimal

$$L := \int_a^b \|\dot{\gamma}(t)\| \,\mathrm{d}t. \tag{7}$$

- THE SOLUTION. (OPTIONAL)

  We have

$$L = \int_a^b \sqrt{1 + \dot{z}(t)^2} \,\mathrm{d}t \tag{8}$$

  with $z(t)$ satisfying $z(a) = z_A$, $z(b) = z_B$ and also $\cos a = x_A$, $\cos b = y_A$; $\sin a = x_B$, $\sin b = y_B$.

Since $\dot{z}(t)$ is a smooth function, we have

$$L = \lim_{k \to \infty} \frac{b-a}{k} \sum_{i=0}^{k-1} \sqrt{1+\dot{z}(t_i)^2} \tag{9}$$

where $t_i = a + \frac{i}{k}(b-a)$. Now notice that the function $f(x) := \sqrt{1+x^2}$ is convex, by Jensen's inequality we have

$$\frac{1}{k} \sum_{i=0}^{k-1} \sqrt{1+\dot{z}(t_i)^2} \geqslant \sqrt{1+\left(\frac{1}{k}\sum_{i=0}^{k-1} \dot{z}(t_i)\right)^2}. \tag{10}$$

As

$$\lim_{k \to \infty} \frac{b-a}{k} \sum_{i=0}^{k-1} \dot{z}(t_i) = \int_a^b \dot{z}(t)\, \mathrm{d}t = z_A - z_B \tag{11}$$

we have

$$\begin{aligned} L &\geqslant (b-a)\sqrt{1+\left(\frac{z_A - z_B}{b-a}\right)^2} \\ &= \sqrt{(b-a)^2 + (z_A - z_B)^2} \\ &\geqslant \sqrt{(b_0-a_0)^2 + (z_A - z_B)^2}. \end{aligned} \tag{12}$$

where $b_0 - a_0 < \pi$ and satisfies also $\cos a_0 = x_A, \cos b_0 = x_B; \sin a_0 = y_A, \sin b_0 = y_B$.

On the other hand, the arc length of the curve

$$\left(\cos t, \sin t, z_A + \frac{t-a_0}{b_0-a_0}(z_B - z_A)\right) \tag{13}$$

is exactly $\sqrt{(b_0-a_0)^2 + (z_B - z_A)^2}$.

Exercise 2. Visualize this shortest path. What would it look like if we "flatten" the cylinder?

3.3. On the sphere

- THE PROBLEM.

  Let A, B be two points on unit sphere centering at the origin. Find the shortest path on the sphere connecting them.

- MATHEMATICAL FORMULATION.

  Among all curves with $\gamma(a) = A$, $\gamma(b) = B$, $x^2(t) + y^2(t) + z^2(t) = 1$, find the one minimizing the integral

$$L := \int_a^b \|\dot{\gamma}(t)\|\, \mathrm{d}t. \tag{14}$$

- THE SOLUTION. (OPTIONAL)

  Our goal is to show that the minimizing curve is the great arc connecting A, B. For arbitrary $\gamma(t)$ on the sphere connecting A, B, we define a new curve:

$$\Gamma(t) = (0, r(t), z(t)) \tag{15}$$

where $r(t) := (x(t)^2 + y(t)^2)^{1/2}$. We notice that $\Gamma(t)$ connects y, z and covers the great arc connecting y, z. Therefore the arc length of $\Gamma(t)$ is no less than $\pi/2$. For $X(t)$ we calculate

$$\begin{aligned} L_\Gamma &= \int_a^b \sqrt{\dot{r}(t)^2 + \dot{z}(t)^2}\, \mathrm{d}t \\ &= \int_a^b \sqrt{\frac{(x(t)\,\dot{x}(t) + y(t)\,\dot{y}(t))^2}{x(t)^2 + y(t)^2} + \dot{z}(t)^2}\, \mathrm{d}t \\ &\leqslant \int_a^b \sqrt{\dot{x}(t)^2 + \dot{y}(t)^2 + \dot{z}(t)^2}\, \mathrm{d}t = L. \end{aligned} \tag{16}$$

Technical Aside

CAUCHY-SCHWARTZ. The crucial step, the one with $\leqslant$, is due to the so-called Cauchy-Schwartz inequality for vectors:

$$|u \cdot v| \leqslant \|u\|\,\|v\|. \tag{17}$$

To prove it, notice that

$$(u - t\,v) \cdot (u - t\,v) = \|u - t\,v\|^2 \geqslant 0 \tag{18}$$

for all t. Expanding the left hand side we see that

$$\|v\|^2\, t^2 - 2\,(u \cdot v)\,t + \|u\|^2 \geqslant 0 \tag{19}$$

holds for all t, and the conclusion follows.

Exercise 3. Finish the proof. (Hint: When is a quadratic equation $t^2 - A\,t + B = 0$ having at most one real solution?)

Remark 1. From the above we see that for each case (plane, cyclinder, sphere), a new idea/technique is needed, worse still, we have to somehow know the answer before we start–much the same as the situation in classical geometry. **The reason for this is that we have used too little calculus**. In the following weeks, we will apply more calculus to geometry problems, and eventually develop a complete theory for the problem “finding shortest path on a surface”. In this theory, the discovery of such path will be reduced to the solution of a single set of ODEs whose derivation is mechanical.

NO MORE AD HOC IDEAS, NO MORE CLEVER TRICKS.