Math 348 Differential Geometry of Curves and Surfaces

Lecture 5 Functions Between Surfaces

Xinwei Yu

Sept. 19, 2017

CAB 527, xinwei2@ualberta.ca
Department of Mathematical & Statistical Sciences
University of Alberta

Table of contents

Please do not hesitate to interrupt me if you have a question.

Brief Review

Last Lecture

- Definitions.
  1. **Surface (Surface Patch, Regular Surface Patch)** [1].
     $\sigma : U \mapsto \mathbb{R}^3$. U: open set in $\mathbb{R}^2$, σ: bijective, both σ and σ^{-1} smooth, and $\sigma_u \times \sigma_v \neq 0$ anywhere.
- Formulas.
  1. Tangent Plane.
     $$\boxed{T_{\sigma(u_0,v_0)}S = \{a\sigma_u(u_0,v_0) + b\sigma_v(u_0,v_0) : a,b \in \mathbb{R}\}}$$
  2. Surface Area.
     $$\boxed{A = \int_U \|\sigma_u \times \sigma_v\| \,\mathrm{d}u\mathrm{d}v.}$$
- Procedures.

[1] Only in this course (348 Fall 2017) that we do not make distinction between them.

Smooth Functions Between Surfaces

Functions Between Surfaces

Setup:

$S, \tilde{S}$: Two surfaces represented by patches $\sigma : U \mapsto \mathbb{R}^3, \tilde{\sigma} : \tilde{U} \mapsto \mathbb{R}^3$

- Functions between surfaces.

  $f : S \mapsto \tilde{S}$.
- Everyday examples.
  - Planar map;
  - Globe.
- Questions to Answer.
  1. How to define smoothness for f?
  2. How to differentiate f?
  3. Geometric interpretations of the derivatives?

Smoothness of Functions Between Surfaces

$$\boxed{S : \sigma : U \mapsto \mathbb{R}^3;\ \tilde{S} : \tilde{\sigma} : \tilde{U} \mapsto \mathbb{R}^3.\ \ f : S \mapsto \tilde{S}.}$$

- **Smoothness.**
  - Note: f may not be defined on the whole $\mathbb{R}^3$.
  - f is said to be smooth if the composite function $F : U \mapsto \tilde{U}$, defined as $F = (\tilde{\sigma})^{-1} \circ f \circ \sigma$, is smooth.

Example

Let S be the upper hemisphere represented by $\sigma(u, v) = (u, v, \sqrt{1 - u^2 - v^2})$ with U the unit disk. Let $\tilde{S}$ be part of the paraboloid $\tilde{\sigma}(\tilde{u}, \tilde{v}) = (\tilde{u} - 1, \tilde{v} - 1, (\tilde{u} - 1)^2 + (\tilde{v} - 1)^2)$ with $\tilde{U}$ also the disk $\{(\tilde{u}, \tilde{v}) \mid (\tilde{u} - 1)^2 + (\tilde{v} - 1)^2 < 1\}$. Let $f(x, y, z) = (x, y, x^2 + y^2)$. Then we have

$$F(u, v) = (u + 1, v + 1).$$

$\boxed{f : S \mapsto \tilde{S}.\ p \in S.\ D_p f\text{: Differential of } f \text{ at } p.}$

- **Meanings of $D_p f$.**
  - Map Analogy: Relation between velocities.

$$D_p f(\text{velocity at } p \in S) = \text{velocity at } f(p) \in \tilde{S}.$$

  - Geometry: Relation between tangent vectors. $D_p f : T_p S \mapsto T_{f(p)} \tilde{S}$;
  - Mathematics: $D_p f$ is a linear transformation from $T_p S$ to $T_{f(p)} \tilde{S}$.

Linear transformation $\Rightarrow$ matrix representation. How to calculate?

`Idea:` Use $F = (\tilde{\sigma})^{-1} \circ f \circ \sigma$.

Calculation of $D_p f$

$$\boxed{f : S \mapsto \tilde{S}.\ p \in S.\ F = (\tilde{\sigma})^{-1} \circ f \circ \sigma.}$$

- **Basis for** T_pS**:** σ_u, σ_v. Calculated at $p = \sigma(u, v)$.
- **Basis for** $T_{f(p)}\tilde{S}$**:** $\tilde{\sigma}_{\tilde{u}}, \tilde{\sigma}_{\tilde{v}}$. Calculated at $f(p) = \tilde{\sigma}(F(u, v))$.
- **Key identity:**

$$\tilde{\sigma}(F(u, v)) = f(\sigma(u, v))$$

.

- **Calculation of** $D_p f$**.**

$$\boxed{\begin{gathered} D_p f(a\sigma_u + b\sigma_v) = \tilde{a}\tilde{\sigma}_{\tilde{u}} + \tilde{b}\tilde{\sigma}_{\tilde{v}}, \\ \begin{pmatrix} \tilde{a} \\ \tilde{b} \end{pmatrix} = DF(u_0, v_0) \cdot \begin{pmatrix} a \\ b \end{pmatrix}. \end{gathered}}$$

Procedures for the Calculation of $D_p f$

To calculate $D_p f$,

1. Pick σ for S covering p (if not given);
2. Pick $\tilde{\sigma}$ for $\tilde{S}$ covering $f(p)$ (if not given);
3. Formulate $F = (\tilde{\sigma})^{-1} \circ f \circ \sigma$;
4. Calculate $DF(u_0, v_0)$ at $\sigma(u_0, v_0) = p$.

$$\boxed{\begin{gathered} D_p f(a\sigma_u + b\sigma_v) = \tilde{a}\tilde{\sigma}_{\tilde{u}} + \tilde{b}\tilde{\sigma}_{\tilde{v}}, \\ \begin{pmatrix} \tilde{a} \\ \tilde{b} \end{pmatrix} = DF(u_0, v_0) \cdot \begin{pmatrix} a \\ b \end{pmatrix}. \end{gathered}}$$

Example

Calculate $D_p f$ for $p = (0, 0, 0)$ and

$$f(u, v, 0) = \left(\frac{4u}{u^2 + v^2 + 4}, \frac{4v}{u^2 + v^2 + 4}, \frac{2(u^2 + v^2)}{u^2 + v^2 + 4} \right).$$

The Gauss Map

- **The Unit Normal.**

$$N(u,v) = \pm \frac{\sigma_u \times \sigma_v}{\|\sigma_u \times \sigma_v\|}$$

  $+$ or $-$? We will not worry about this in 348 Fall 2017. From now on we simply take $+$.
- **The Gauss Map.**
  - $\mathcal{G} : S \mapsto \mathbb{S}^2$, $\mathcal{G}(p) = N(u_0, v_0)$ where $p = \sigma(u_0, v_0)$.
- **Intuition of the Gauss Map.**

Example

1. Plane;
2. Cylinder
3. Sphere;
4. Paraboloid.

Properties of the Gauss Map

- Visualization: Catenoid, Cylinder.
- Natural surface patch: $N(u, v)$.
- $T_pS = T_{f(p)}\mathbb{S}^2$.
- Two "natural" basis for $T_{f(p)}\mathbb{S}^2$.
    1. $\{N_u, N_v\}$;
    2. $\{\sigma_u, \sigma_v\}$.
- Matrix representation for $D_p\mathcal{G}$ using either basis.

Looking Back and Forward

Summary

- **Required Textbook Sections.** §4.1–4.4, §4.5 (before Definition 4.5.1).
- **Optional Textbook Sections.** Rest of §4.5, §5.1–5.6.

- **Definitions.**
    1. f is smooth $\iff F = (\tilde{\sigma})^{-1} \circ f \circ \sigma$ is smooth.
    2. The Gauss Map $\mathcal{G} : S \mapsto \mathbb{S}^2$, $\mathcal{G}(p) = N(u_0, v_0)$ where $p = \sigma(u_0, v_0)$.
- **Formulas.**
    1. Differential.
    $$\begin{pmatrix} \tilde{a} \\ \tilde{b} \end{pmatrix} = DF(u_0, v_0) \cdot \begin{pmatrix} a \\ b \end{pmatrix}.$$
    2. Unit normal.
    $$N(u, v) = \frac{\sigma_u \times \sigma_v}{\|\sigma_u \times \sigma_v\|}.$$
- **Procedures.**

    Calculate $D_p f$.

    1. Pick σ for S covering p (if not given);
    2. Pick $\tilde{\sigma}$ for $\tilde{S}$ covering $f(p)$ (if not given);
    3. Formulate $F = (\tilde{\sigma})^{-1} \circ f \circ \sigma$;
    4. Calculate $DF(u_0, v_0)$ at $\sigma(u_0, v_0) = p$.

See You Thursday!

Isometry

- What is isometry?
- Properties of isometry.
- Isometry and maps.
- No map for the sphere.