Lecture 5: Surfaces II: Functions Between Surfaces

Disclaimer. As we have a textbook, this lecture note is for guidance and supplement only. It should not be relied on when preparing for exams.

In this lecture we give mathematical definition of surfaces as a compatible collection of surface patches. We also define the tangent plane and normal vectors of surfaces.

The required textbook sections are §4.2, §4.3, §4.4.

The examples in this note are mostly different from examples in the textbook. Please read the textbook carefully and try your hands on the exercises. During this please don't hesitate to contact me if you have any questions.

Table of contents

Lecture 5: Surfaces II: Functions Between Surfaces 1

1. Functions between surfaces 2
2. The Gauss Map 4

1. Functions between surfaces

The transition from multivariable calculus to classical differential geometry is fulfilled when we start to differentiate functions mapping one surface to another. Such differentiation is defined through the help of surface patches. The differentials are linear maps between tangent planes.

- Consider two surfaces $S, \tilde{S}$. We can consider a function f from S to $\tilde{S}$, that is given $p \in S$, we have $\tilde{p} = f(p) \in S'$ defined.[1]
- Let $f: S \mapsto \tilde{S}$ be a function from one surface S to another surface $\tilde{S}$. Let $p \in S$. We would like to "differentiate" f at p.
- $D_p f: T_p S \mapsto T_{\tilde{p}} \tilde{S}$ is a linear map. Now note that
    - Any vector in $T_p S$ can be written as $a\,\sigma_u(p) + b\,\sigma_v(p)$.
    - Any vector in $T_{\tilde{p}} \tilde{S}$ can be written as $\tilde{a}\,\tilde{\sigma}_{\tilde{u}}(\tilde{p}) + \tilde{b}\,\tilde{\sigma}_{\tilde{v}}(\tilde{p})$. Recall that $\tilde{p} = f(p)$.

  Thus if M is the matrix representation of $D_p f$ with respect to these bases, there holds

$$D_p f(a\,\sigma_u(p) + b\,\sigma_v(p)) = \tilde{a}\,\tilde{\sigma}_{\tilde{u}}(\tilde{p}) + \tilde{b}\,\tilde{\sigma}_{\tilde{v}}(\tilde{p}) \Longleftrightarrow \begin{pmatrix} \tilde{a} \\ \tilde{b} \end{pmatrix} = M \begin{pmatrix} a \\ b \end{pmatrix}. \tag{1}$$

- The procedure.
    i. Let $\sigma: U \mapsto \mathbb{R}^3$ be a surface patch of S covering p: $\sigma(u_0, v_0) = p$.
    ii. Let $\tilde{\sigma}: \tilde{U} \mapsto \mathbb{R}^3$ be a surface patch of $\tilde{S}$ covering $f(p)$.
    iii. Let $F := (\tilde{\sigma})^{-1} \circ f \circ \sigma: U \mapsto \tilde{U}$.
    iv. We have

$$D_p f(a\,\sigma_u(p) + b\,\sigma_v(p)) = \tilde{a}\,\tilde{\sigma}_{\tilde{u}}(\tilde{p}) + \tilde{b}\,\tilde{\sigma}_{\tilde{v}}(\tilde{p}) \Longleftrightarrow \begin{pmatrix} \tilde{a} \\ \tilde{b} \end{pmatrix} = DF(u, v) \begin{pmatrix} a \\ b \end{pmatrix}. \tag{2}$$

    Here if $F(u, v) = (F_1(u, v), F_2(u, v))$, the matrix

$$DF = \begin{pmatrix} F_{1,u} & F_{1,v} \\ F_{2,u} & F_{2,v} \end{pmatrix}. \tag{3}$$

- Proof of the formula (2).
    1. Let $\gamma(t) := \sigma(u(t), v(t))$ be a curve on S with $\gamma(t_0) = p$. Let $u'(t_0) = a, v'(t_0) = b$. We see that

$$\dot{\gamma}(t_0) = a\,\sigma_u(p) + b\,\sigma_v(p). \tag{4}$$

    2. Consider the curve $\tilde{\gamma}(t) := (\tilde{u}(t), \tilde{v}(t)) = F(u(t), v(t))$. Then the chain rule gives

$$\begin{pmatrix} \tilde{u}'(t_0) \\ \tilde{v}'(t_0) \end{pmatrix} = DF(u(t_0), v(t_0)) \cdot \begin{pmatrix} u'(t_0) \\ v'(t_0) \end{pmatrix} = DF(u(t_0), v(t_0)) \cdot \begin{pmatrix} a \\ b \end{pmatrix} \tag{5}$$

1. For example, the correspondence between a map and the real locations is such a function.

3. Finally notice that if $\gamma(t)$ is a curve on S, with $\gamma(t_0) = p$, then $\tilde{\gamma}(t) := f(\gamma(t))$ is a curve on $\tilde{S}$, with $\tilde{\gamma}(t_0) = f(p) = \tilde{p}$, and furthermore

$$\dot{\tilde{\gamma}}(t_0) = D_p f(\dot{\gamma}(t_0)). \tag{6}$$

Consequently $\tilde{a} = \tilde{u}'(t_0), \tilde{b} = \tilde{v}'(t_0)$.

Summary.

The differential of $f: S \mapsto \tilde{S}$ at $p \in S$, denoted $D_p f$, is a linear map between the tangent planes $T_p S$ and $T_{f(p)} \tilde{S}$. If σ and $\tilde{\sigma}$ are two surface patches on $S, \tilde{S}$ respectively, containing p, $f(p)$ respectively, then the matrix representation of $D_p f$ is the 2×2 Jacobian matrix $DF(u_0, v_0)$ where $F := (\tilde{\sigma})^{-1} \circ f \circ \sigma$, and $\sigma(u_0, v_0) = p$. In other words, we have

$$D_p f(a\,\sigma_u + b\,\sigma_v) = \tilde{a}\,\tilde{\sigma}_u + \tilde{b}\,\tilde{\sigma}_v \tag{7}$$

where all the σ_u, σ_v are evaluated at p and $\tilde{\sigma}_u, \tilde{\sigma}_v$ at $f(p)$, and

$$\begin{pmatrix} \tilde{a} \\ \tilde{b} \end{pmatrix} = DF(u_0, v_0) \cdot \begin{pmatrix} a \\ b \end{pmatrix}. \tag{8}$$

Example 1. (Stereographic projection) Let $\tilde{S}$ be the sphere $x_1^2 + x_2^2 + (x_3 - 1)^2 = 1$ taking away the north pole $(0, 0, 2)$. Let S be the plane $x_3 = 0$. $f: S \mapsto \tilde{S}$ be such that $(0, 0, 2)$, $(u, v, 0)$, $f(u, v, 0)$ lie on the same straight line. Then we have

$$f(u, v, 0) = \left(\frac{4\,u}{u^2 + v^2 + 4}, \frac{4\,v}{u^2 + v^2 + 4}, \frac{2\,(u^2 + v^2)}{u^2 + v^2 + 4} \right). \tag{9}$$

Let $p = (0, 0, 0)$. We will calculate $D_p f$.

i. Pick σ covering p. We take $\sigma: U = \mathbb{R}^2 \mapsto \mathbb{R}^3$ defined as $\sigma(u, v) = (u, v, 0)$;

ii. Pick $\tilde{\sigma}$ covering $f(p)$ and calculate $(\tilde{\sigma})^{-1}$. We calculate $f(p) = (0, 0, 0)$. Thus we can take

$$\tilde{\sigma}: \tilde{U} = \{\tilde{u}^2 + \tilde{v}^2 < 1\} \mapsto \mathbb{R}^3, \qquad \tilde{\sigma}(\tilde{u}, \tilde{v}) = \left(\tilde{u}, \tilde{v}, 1 - \sqrt{1 - \tilde{u}^2 - \tilde{v}^2}\right). \tag{10}$$

Thus $(\tilde{\sigma})^{-1}(x, y, z) = (x, y)$.

iii. Formulate $F = (\tilde{\sigma})^{-1} \circ f \circ \sigma$. We have

$$F = (\tilde{\sigma})^{-1} \circ f: F(u, v) = \left(\frac{4\,u}{u^2 + v^2 + 4}, \frac{4\,v}{u^2 + v^2 + 4} \right). \tag{11}$$

iv. Calculate $DF(u_0, v_0)$ for $\sigma(u_0, v_0) = p$. We have $\sigma(0, 0) = (0, 0, 0) = p$ and therefore calculate

$$DF(0, 0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{12}$$

The conclusion from the above calculation is that, if $v = a\,\sigma_u + b\,\sigma_v$ is a vector in $T_p S$, then $D_p f(v) \in T_{f(p)} \tilde{S}$ is given by $a\,\tilde{\sigma}_u + b\,\tilde{\sigma}_v$.

Exercise 1. Calculate DF at a different point.

2. The Gauss Map

- NORMAL VECTOR.

  DEFINITION 2. (NORMAL VECTOR) *A normal vector at $p \in S$ is a vector that is perpendicular to all tangent vectors at p. A unit normal vector at $p \in S$ is a normal vector at p with unit norm.*

  Let $U \subseteq \mathbb{R}^2$ and $\sigma: U \mapsto \mathbb{R}^3$ be a surface patch of a surface S. Let $p = \sigma(u_0, v_0)$ for some $(u_0, v_0) \in U$. Then the normal vectors at p are given by $c\,\sigma_u \times \sigma_v$ where $c \in \mathbb{R}$. In particular, the unit normal vectors are given by

  $$\pm \frac{\sigma_u \times \sigma_v}{\|\sigma_u \times \sigma_v\|}. \tag{13}$$

- ORIENTATION.

  - INFORMAL DEFINITION. A surface S is orientable if and only if there is a continuous function $N: S \mapsto \mathbb{R}^3$ such that at every $p \in S$, $N(p)$ is a unit normal vector of S at p .
  - There are surfaces that are not orientable.
  - Every regular surface patch is orientable.

- THE GAUSS MAP.

  DEFINITION 3. (GAUSS MAP) *The Gauss map $\mathcal{G}: S \mapsto \mathbb{S}^2$ is defined as $\mathcal{G}(p) = N(p)$, the unit normal of the surface S at point $p \in S$.*

- Calculation of the Gauss map at $p \in S$.

  1. Let σ be the surface patch map for S. Let $\sigma(u_0, v_0) = p$.
  2. Calculate
  $$\mathcal{G}(p) = N(u_0, v_0) = \frac{\sigma_u(u_0, v_0) \times \sigma_v(u_0, v_0)}{\|\sigma_u(u_0, v_0) \times \sigma_v(u_0, v_0)\|}. \tag{14}$$

  Example 4. Let S be the paraboloid $z = x^2 + y^2$. Calculate $\mathcal{G}(p)$ for $p = (1, 1, 2)$.

  Solution.

  1. We take $\sigma(u, v) = (u, v, u^2 + v^2)$. We have $\sigma(1, 1) = (1, 1, 2) = p$.
  2. Calculate

  $$\begin{aligned} \sigma_u &= (1, 0, 2\,u) \Longrightarrow \sigma_u(1, 1) = (1, 0, 2); \\ \sigma_v &= (0, 1, 2\,v) \Longrightarrow \sigma_v(1, 1) = (0, 1, 2); \\ \sigma_u(1, 1) \times \sigma_v(1, 1) &= (-2, -2, 1). \end{aligned}$$

Thus

$$\mathcal{G}(1,1,2) = \left(-\frac{2}{3}, -\frac{2}{3}, 1\right). \tag{15}$$