Math 348 Differential Geometry of Curves and Surfaces

Lecture 4 Surfaces in Calculus

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Please do not hesitate to interrupt me if you have a question.

How to Define a Surface Mathematically?

Parametrization

- Level Set.
  - 3D. $f(x, y, z) = 0$.
  - nD?
- Parametrization.
  - Motivation: A ”distorted” piece of part of the 2D plane[1].
  - 3D. $(x(u, v), y(u, v), z(u, v))$.
  - nD?
- What About Graphs $z = f(x, y)$?
  - Both ”level set” and ”parametrization”.

[1] Deforming a map on a elastic membrane into the shape of the real terrain.

Pros and Cons

- Level Set Representations.
  - Includes ”non-surfaces”[2].
  - In-efficient for ”local” theoretical study.
  - Efficient for ”global” theoretical study.
  - Efficient computational study (Level Set Method).
- Parametrization.
  - May not be able to represent the whole surface faithfully[3].
  - Includes ”non-surfaces”.
  - In-efficient for computational study.
  - Efficient for ”local” theoretical study.
- **Our Choice.** Parametrization.

[2] Whitney’s Theorem again!

[3] Either not one-to-one or not covering the whole surface.

Surface Patch

- **Surface Patch.**

  A surface patch is a function $\sigma : U \mapsto \mathbb{R}^3$, where U is an open set in $\mathbb{R}^2$, σ is bijective, and both σ and its inverse σ^{-1} are continuous.
- **Intuition.**

  A surface patch is a "map"[4].
- **Regular Surface Patch.**

  A regular surface patch is a surface patch σ that is smooth and furthermore satisfying $\sigma_u \times \sigma_v \neq 0$ anywhere in U.
- **Surface Patch vs Surface.**
  - One surface patch may not be enough to "cover" the whole surface. (Example: The unit sphere.)
  - Will focus on surface patches in 348.

In the remaining of the course, "surface" = "a regular surface patch".

[4] Think those ancient maps, drawn on a piece of parchment, with irregular boundary. Like this one.

Example

A torus (Wiki).

$$\sigma(u,v) = ((a + b\cos u)\cos v, (a + b\cos u)\sin v, b\sin u),$$

$(u,v) \in U = (0,2\pi) \times (0,2\pi)$. We require that $0 < b < a$.

Example

A helicoid (Wiki).

$$\sigma(u,v) = (v\cos u, v\sin u, \lambda u),$$

$(u,v) \in U = (-\infty,\infty) \times (0,\infty)$, where $\lambda \in \mathbb{R}$ is a constant.

Note that the image of σ in the torus example does not "cover" the "full torus". We will not emphasize on this issue in 348, though it will be come importat in the study of manifolds.

Differentiation

- **First Order Derivatives.**
  - Mathematics.
    - $\sigma(u,v) : (x(u,v), y(u,v), z(u,v)), (u,v) \in U$.
    - $\sigma_u := \frac{\partial \sigma}{\partial u}, \sigma_v := \frac{\partial \sigma}{\partial v}$.
  - Map Analogy:

    σ is the relation between the map U and the real world region it represents. When a car at $\sigma(u_0, v_0)$ moves with velocity $\sigma_u(u_0, v_0)$, its avatar in the map, at location (u_0, v_0), moves with speed 1 in the u direction, that is moves with velocity $(1, 0)$.
  - Geometry:

    $\sigma_u(u_0, v_0)$ and $\sigma_v(u_0, v_0)$ are two vectors tangent to the surface at $\sigma(u_0, v_0)$.

    σ is regular $\iff \sigma_u \times \sigma_v \neq 0 \iff \sigma_u$ and σ_v span the "tangent plane".
- **Higher Order Derivatives.**

  What are the geometrical meaning of $\sigma_{uu}, \sigma_{uv}, \sigma_{vv}$? Answering this question is one of the main goals of classical DG.

Examples

S: a surface represented by σ; $p = \sigma(u_0, v_0) \in S$. The tangent plane at p is

$$\boxed{T_pS = \{a\sigma_u(u_0, v_0) + b\sigma_v(u_0, v_0) : a, b \in \mathbb{R}\}.}$$

Example

Consider the torus

$$\sigma(u, v) = ((2 + \cos u)\cos v, (2 + \cos u)\sin v, \sin u).$$

We calculate

$$\sigma_u = (-\sin u \cos v, -\sin u \sin v, \cos u)$$
$$\sigma_v = (-(2 + \cos u)\sin v, (2 + \cos u)\cos v, 0).$$

Now at $p = (-1, 0, 0) = \sigma(\pi, \pi)$, we have the tangent plane

$$T_pS = \{a(0, 0, -1) + b(0, -1, 0) : a, b \in \mathbb{R}.\} = \text{The } yz \text{ plane}$$

How to Measure on a Surface

The Surface Area Formula

$$\boxed{A = \int_U \|\sigma_u \times \sigma_v\| \, \mathrm{d}u\mathrm{d}v.}$$

- **Motivation.**
    - $\sigma(u, v) = u(a_1, a_2, a_3) + v(b_1, b_2, b_3)$, $U = (0, 1)^2$;
    - $\sigma_u(u, v) = (a_1, a_2, a_3)$, $\sigma_v(u, v) = (b_1, b_2, b_3)$;
    - Area of $\sigma(U)$ is $\|(a_1, a_2, a_3) \times (b_1, b_2, b_3)\|$.
- **Natural Idea:** Approximate by triangles?
- **Mathematical Definition.**
    - Subtle: Schwarz's Lantern[5].
    - Can be fixed for smooth surfaces[6].

[5] Demonstration, Origami. Even experts in geometry was surprised.
[6] See e.g. my lecture note for Math 317

Example

1. Unit sphere.
    - Hemisphere;
    - Spherical coordinates.
2. The torus $\sigma(u, v) = ((2 + \cos u)\cos v, (2 + \sin u)\sin v, \sin u)$.
3. The area of the part of $z = xy$ that is inside $x^2 + y^2 = 1$.

Looking Back and Forward

Summary

- **Required Textbook Sections.** §4.1, §4.2, §4.3, §4.4, §4.5 (before Definition 4.5.1).
- **Optional Textbook Sections.** Rest of §4.5, §5.1–5.6.

- **Definitions.**
  1. Regular Surface Patch.
     $\sigma : U \mapsto \mathbb{R}^3$. U: open set in $\mathbb{R}^2$, σ: bijective, both σ and σ^{-1} smooth, and $\sigma_u \times \sigma_v \neq 0$ anywhere.
- **Formulas.**
  1. Tangent Plane.
     $$\boxed{T_{\sigma(u_0,v_0)}S = \{a\sigma_u(u_0,v_0) + b\sigma_v(u_0,v_0) : a, b \in \mathbb{R}\}}$$
  2. Surface Area.
     $$\boxed{A = \int_U \|\sigma_u \times \sigma_v\| \, \mathrm{d}u\mathrm{d}v.}$$
- **Procedures.**

See You Tuesday!

Functions Between Surfaces

- Functions between surfaces and the geometrical meanings of their derivatives;
- The most important function in 348: Gauss map.
- Isometry.