Lecture 4: Surfaces I: Surfaces in Calculus

Disclaimer. As we have a textbook, this lecture note is for guidance and supplement only. It should not be relied on when preparing for exams.

In this lecture we give mathematical definition of surfaces as a compatible collection of surface patches. We also define the tangent plane and present the surface area formula.

The required textbook sections are §4.1--4.4, §4.5 (before Definition 4.5.1). The optional textbook sections are §4.5 (Definition 4.5.1 and after), §5.1–5.6.

The examples in this note are mostly different from examples in the textbook. Please read the textbook carefully and try your hands on the exercises. During this please don't hesitate to contact me if you have any questions.

Table of contents

Lecture 4: Surfaces I: Surfaces in Calculus 1

1. Parametrization of Surfaces 2
   1.1. The difficulties in defining surfaces mathematically 2
   1.2. Surfaces 3
   1.3. Tangent planes 5
2. Surface Area 6
   2.1. How to calculate surface area 6
   2.2. The counterexample of Schwartz (optional) 8

1. Parametrization of Surfaces

Mathematical representation of an arbitrary smooth surface is non-trivial. We need to "break it up" into simple graph-like pieces, called surface patches, and then "glue" these pieces together.

1.1. The difficulties in defining surfaces mathematically

- Two naive definitions.
  - A surface is the graph of a "nice" function.

    Example 1. The graph of $f(x,y)=x^2+y^2$ defines a paraboloid.

    Example 2. It is awkward to define the unit sphere this way.

    Remark 3. This definition is too narrow.

  - A surface is the level set of a "nice" function.

    Exercise 1. Let $f(x,y)$ be a smooth function. Then there is a smooth function $F(x,y,z)$ such that the graph of $f(x,y)$ is the zero levelset $\{(x,y,z) : F(x,y,z)=0\}$.

    Example 4. The unit sphere is $f(x,y,z)=0$ for $f(x,y,z)=x^2+y^2+z^2-1$.

    However, a direct consequence of Whitney's extension theorem[1] is that, any closed set in $\mathbb{R}^n$ is the zero level set of a smooth function $f(x_1,\ldots,x_n)$, that is for every closed set A, there is a smooth function f such that $A=\{f=0\}$.

1. Whitney, Hassler (1934), "Analytic extensions of functions defined in closed sets", Transactions of the American Mathematical Society, American Mathematical Society, 36 (1): 63–89, doi:10.2307/1989708.

Technical Aside

- Closed set. A set $A \subseteq \mathbb{R}^n$ is closed if and only if it is the complement of an open set in $\mathbb{R}^n$.
  - Complement of a set. The complement of a set A is another set consisting of all points that are not in A. We usually denote this set by A^c.
- Open set. A set $A \subseteq \mathbb{R}^n$ is open if and only if it is the union of open balls.
  - Union of sets. The union of a collection $\mathcal{W}$ of sets is another set consisting of all points that belong to at least one set in the collection. We denote the new set by $\cup_{A \in \mathcal{W}} A$.

    Exercise 2. Determine $\cup_{i=1}^{\infty}(i, i+1)$.

    Exercise 3. Determine $\cup_{k=1}^{\infty}\left(1-\frac{1}{k}, 1-\frac{1}{k+1}\right)$.
  - Open ball. An "open ball" in $\mathbb{R}^n$ is the set of all points $x \in \mathbb{R}^n$ satisfying $\|x - x_0\| < r$ for some $x_0 \in \mathbb{R}^n$ and $r > 0$.

  Exercise 4. The union of open sets is still an open set.

Example 5. (Cantor set) Consider the following set F obtained through an infinite process:

i. Take the closed interval $[0, 1]$. Drop the middle third $(1/3, 2/3)$.
ii. Take the remaining set $[0, 1/3] \cup [2/3, 1]$, drop third middle third $(1/9, 2/9), (7/9, 8/9)$.
iii. Repeat this ad infinitum.

The remaining points form a infinite closed set.

Exercise 5. Convince yourself that the Cantor set is infinite and closed.

Example 6. (Sierpinski carpet) A Sierpinski carpet is an analog of Cantor set in $\mathbb{R}^2$. We start from the unit square and repreatedly take away the middle $1/9$.

Exercise 6. Describe the process more precisely.

What remains definitely does not fit our intuition of a "smooth surface", but it is the level set of a smooth function by Whitney's theorem.

Remark 7. This definition is too wide.

1.2. Surfaces

- Surface patches.
  - The generalization of "graph".
    - Motivation from curves.

We used to think of curves as graphs of a single variable function $f(x)$, but later generalize it to the "trace" of a vector function $(x_1(t), \ldots, x_n(t))$.

- Generalization of graph.

  Instead of considering the graph of a two variable function $f(x, y)$, we consider the "trace" of the vector function $\sigma(u, v) = (x(u, v), y(u, v), z(u, v))$.

- Surface patch.

DEFINITION 8. (SURFACE PATCH)
A surface patch is a function $\sigma: U \mapsto \mathbb{R}^3$ such that both σ and its inverse σ^{-1} are continuous, and σ is bijective.

Technical Aside

- Such a function f is called a homeomorphism.
- Bijective. A function $f: X \mapsto Y$ is called bijective if
  i. f is one-to-one. That is $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$;
  ii. f is onto. That is for every $y \in Y$ there is $x \in X$ such that $f(x) = y$.
- Inverse function. When a function is bijective, we see that the mapping $y \mapsto$ the particular x such that $f(x) = y$ is also a function. We call it the inverse function of f. Denoted f^{-1}. We note that $f^{-1}: Y \mapsto X$.

Exercise 7. What goes wrong if f is not bijective?

Remark 9. A surface patch is a "mathematical deformation" of a piece of the flat plane into a curves surface in space.

Exercise 8. A graph of a function is a surface patch.

- SURFACES.

DEFINITION 10. (SURFACE) *A subset S of $\mathbb{R}^3$ is a surface if, for every point $p \in S$, there is an open set U in $\mathbb{R}^2$ and a surface patch σ from $U \subseteq \mathbb{R}^2$ to S such that $p \in \sigma(U)$.*

Exercise 9. Compare with Definition 4.1.1 of Textbook.

Remark 11. Thus a surface is a subset of $\mathbb{R}^3$ that can be covered with a collection of surface patches. Such a collection is called an atlas of S.

- EXAMPLES OF SURFACES.

Example 12. (GRAPH) Let $U \subseteq \mathbb{R}^2$ and $f: U \mapsto \mathbb{R}$ be a smooth function. Then its graph $\{x_3 = f(x_1, x_2)\}$ is a surface.

Proof. Consider the surface patch $(u, v) \mapsto (u, v, f(u, v))$. $\square$

Example 13. (SPHERE) See textbook Example 4.1.4.

Example 14. (SURFACE OF REVOLUTION) Let $f: (a, b) \mapsto \mathbb{R}$ be a smooth positive function. Its surface of revolution (around the x-axis) is defined as

$$\{(x, y, z): y^2 + z^2 = f(x)^2\}. \tag{1}$$

To see that it is a surface, consider the atlas consisting of two surface patches:

$$(u, f(u)\cos v, f(u)\sin v), \qquad (u, v) \in U := (a, b) \times (0, 2\pi) \tag{2}$$

and

$$(u, f(u)\cos v, f(u)\sin v), \qquad (u, v) \in U := (a, b) \times (-\pi, \pi). \tag{3}$$

Exercise 10. What about the surface of revolution obtained from rotating the graph of f around the y-axis? Is it a surface? Why?

Example 15. (LEVEL SURFACES) Let $f(x, y, z)$ be a smooth function. We have seen that its level surface $S := \{(x, y, z): f(x, y, z) = 0\}$ may not be a smooth surface in any reasonable sense. However we have the following result.

(THEOREM 5.1.1 OF TEXTBOOK) Assume $\nabla f(x, y, z) \neq 0$ for every $(x, y, z) \in S$. Then S is a smooth surface.

1.3. Tangent planes

- TANGENT VECTOR AND TANGENT PLANE.

DEFINITION 16. (DEFINITION 4.2.1 OF TEXTBOOK) *A surface patch $\sigma: U \mapsto \mathbb{R}^3$ is called regular if it is smooth and the vectors σ_u and σ_v are linearly independent at all points $(u, v) \in U$.*

In the following we will always assume the surface under study to have an atlas of regular surface patches. In fact, most of the times we will just focus on one single surface patch.

DEFINITION 17. (DEFINITION 4.4.1 OF TEXTBOOK) *A tangent vector to a surface S at point $p \in S$ is a tangent vector at p of a curve in S passing through p.*

When we consider all the curves in S passing through p, we obtain a collection of tangent vectors. This collection (together with the zero vector) forms a two-dimensional linear vector space called "tangent plane" of S at p. Denoted T_pS.

Exercise 11. Prove that if u, v are tangent vectors at p and a, b are arbitrary real numbers, then $a\,u + b\,v$ is also a tangent vector at p .

PROPOSITION 18. (PROPOSITION 4.4.2) *Let $\sigma: U \mapsto \mathbb{R}^3$ be a patch of a surface S containing a point $p \in S$, and let (u, v) be coordinates in U. The tangent space to S at p is the vector subspace of $\mathbb{R}^3$ spanned by the vectors σ_u and σ_v (the derivatives are evaluated at the point $(u_0, v_0) \in U$ such that $\sigma(u_0, v_0) = p$).*

Remark 19. In other words, we can represent the collection of tangent vectors at p as $\{a\,\sigma_u + b\,\sigma_v: a, b \in \mathbb{R}\}$.

Let $U \subseteq \mathbb{R}^2$ and $\sigma: U \mapsto \mathbb{R}^3$ be a surface patch of a surface S. Let $p = \sigma(u_0, v_0)$ for some $(u_0, v_0) \in U$. Then the tangent plane $T_p S = \{a\, \sigma_u(u_0, v_0) + b\, \sigma_v(u_0, v_0): a, b \in \mathbb{R}\}$.

- Examples.

Example 20. (Graph) Let $U \subseteq \mathbb{R}^2$ and $f: U \mapsto \mathbb{R}$ be a smooth function. Then its graph $\{x_3 = f(x_1, x_2)\}$ is a surface. It is given by one surface patch $(u, v, f(u, v))$. As a consequence, we have

$$\sigma_u = (1, 0, f_u), \qquad \sigma_v = (0, 1, f_v), \tag{4}$$

and the tangent plane $T_p S$ at $p = (u_0, v_0, f(u_0, v_0))$ is given by

$$\{a\,(1, 0, f_u) + b\,(0, 1, f_v): a, b \in \mathbb{R}\}. \tag{5}$$

2. Surface Area

The definition of surface area is subtle. However for the regular surfaces considered in 348, there is a simple formula.

2.1. How to calculate surface area

Intuitions about the surface area formula.

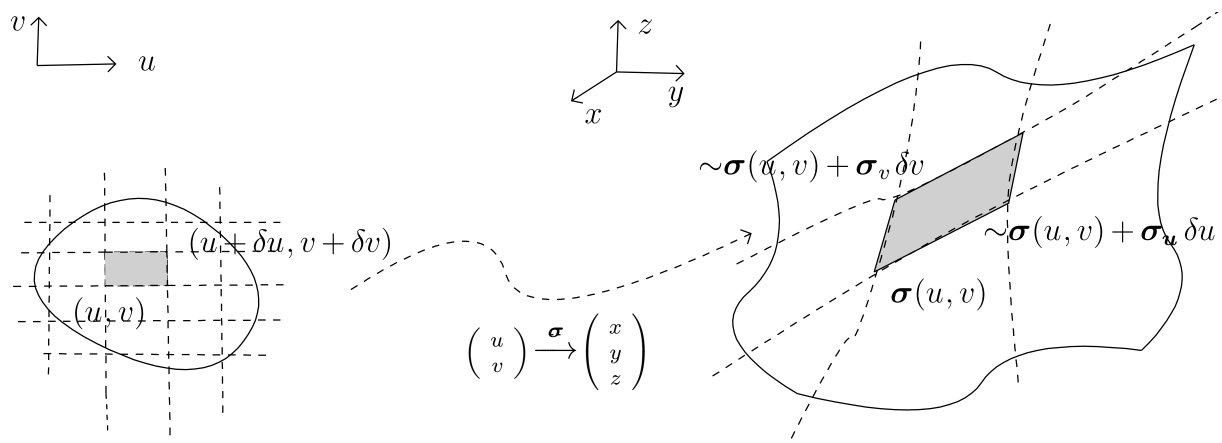

Figure 1. Stretching and twisting of of infinitesimal rectangles.

The shaded rectangle in the (u, v)-plane, with area $\delta u \cdot \delta v$, is "stretched" by the mapping $\boldsymbol{r}$ to the shaded curvilinear parallelogram in the (x, y, z)-space. The sides of this parallelogram are approximately $\boldsymbol{r}_u\, \delta u$ and $\boldsymbol{r}_v\, \delta v$, giving its area to be about $\|\sigma_u \times \sigma_v\|\, \delta u \cdot \delta v$. Summing the areas of all such curvilinear parallelograms up we reach the integral formula

$$\int_U \|\sigma_u \times \sigma_v\|\, \mathrm{d}u \mathrm{d}v. \tag{6}$$

Area of a surface patch:

$$\int_U \|\sigma_u \times \sigma_v\| \, \mathrm{d}u \mathrm{d}v \tag{7}$$

In particular, when the surface patch is given by a graph $z = \phi(x, y)$ on $U \subset \mathbb{R}^2$. Then

$$S = \int_U \sqrt{1 + \phi_x^2 + \phi_y^2} \, \mathrm{d}x \, \mathrm{d}y. \tag{8}$$

Example 21. Find the area of the part of $z = x\,y$ that is inside $x^2 + y^2 = 1$.

Solution. We calculate

$$S = \int_{x^2+y^2 \leqslant 1} \sqrt{1 + z_x^2 + z_y^2} \, \mathrm{d}(x, y) = \frac{2\,\pi}{3} \left(2\sqrt{2} - 1\right). \tag{9}$$

Example 22. Find the surface area of the sphere $x^2 + y^2 + z^2 = R^2$.

Solution. We use the parametrization

$$\sigma(\phi, \psi) = \begin{pmatrix} R\cos\phi\cos\psi \\ R\sin\phi\cos\psi \\ R\sin\psi \end{pmatrix}, \qquad U = \left\{(\phi, \psi) | \, 0 < \phi < 2\,\pi, -\frac{\pi}{2} < \psi < \frac{\pi}{2}\right\}. \tag{10}$$

Exercise 12. Note that as shown on page 72 of the textbook, one "slit" on the sphere is not covered. Convince yourself that this is not a problem for the purpose of calculating surface area. Prove that this is not a problem if you have learned the theory of Riemann integration on surfaces.

Then calculate

$$\sigma_u = \begin{pmatrix} -R\sin\phi\cos\psi \\ R\cos\phi\cos\psi \\ 0 \end{pmatrix}, \qquad \sigma_v = \begin{pmatrix} -R\cos\phi\sin\psi \\ -R\sin\phi\sin\psi \\ R\cos\psi \end{pmatrix}. \tag{11}$$

This gives

$$S = \int_D R^2 \cos\psi \, \mathrm{d}(\phi, \psi) = 4\,\pi\,R^2. \tag{12}$$

Example 23. Find the surface are of the torus $\sigma(u, v) = ((2 + \cos u)\cos v, (2 + \cos u)\sin v, \sin u)$, $u, v \in (0, 2\,\pi)$.

Solution. We calculate

$$\sigma_u = (-\sin u\cos v, -\sin u\sin v, \cos u), \qquad \sigma_v = (-(2 + \cos u)\sin v, (2 + \cos u)\cos v, 0). \tag{13}$$

Now

$$\sigma_u \times \sigma_v = -(2 + \cos u)\,(\cos u\cos v, \cos u\sin v, \sin u) \tag{14}$$

which leads to

$$\|\sigma_u \times \sigma_v\| = 2 + \cos u \tag{15}$$

and

$$\begin{aligned} A &= \int_0^{2\pi} \int_0^{2\pi} (2 + \cos u) \, \mathrm{d}u \, \mathrm{d}v \\ &= 8\,\pi. \end{aligned} \tag{16}$$

2.2. The counterexample of Schwartz (optional)

"The example of Schwarz, ... , was the starting point of an extensive and fascinating literature. Still, we do not possess as yet a satisfactory theory of the area of surfaces, ..."

—— Tibor Rado, 1943[2]

- Gelbaum, B. R. and Olmsted, J. M. H., *Counterexamples in Analysis*, Chapter 11, Example 7.

Let

$$S = \{(x, y, z) | x^2 + y^2 = 1, \quad 0 \leqslant z \leqslant 1\}. \tag{17}$$

Let $m \in \mathbb{N}$. Define $2m + 1$ circles:

$$C_{k,m} := S \cap \left\{(x, y, z) | z = \frac{k}{2m}\right\}, \qquad k = 0, 1, 2, \ldots, 2m. \tag{18}$$

Now let $n \in \mathbb{N}$. Pick on each $C_{k,m}$ n points:

$$P_{k,m,j} := \begin{cases} \left(\cos\frac{2j\pi}{n}, \sin\frac{2j\pi}{n}, \frac{k}{2m}\right) & k \text{ even} \\ \left(\cos\frac{(2j+1)\pi}{n}, \sin\frac{(2j+1)\pi}{n}, \frac{k}{2m}\right) & k \text{ odd} \end{cases}, \qquad j = 0, 1, \ldots, n-1. \tag{19}$$

Connecting this points in a natural manner we obtain $4mn$ congruent space triangles. It can be calculated that the area of each triangle is

$$\sin\left(\frac{\pi}{n}\right)\left[\frac{1}{4m^2} + \left(1 - \cos\left(\frac{\pi}{n}\right)\right)^2\right]^{1/2}. \tag{20}$$

Exercise 13. Prove the above formula.

Thus the area of the polyhedron inscribed in the cylinder is

$$A_{mn} := 2\pi\frac{\sin(\pi/n)}{\pi/n}\left(1 + 4m^2\left(1 - \cos\frac{\pi}{n}\right)^2\right)^{1/2}. \tag{21}$$

Exercise 14. Prove that, as $m, n \to \infty$,

a) the diameters of the triangles $\longrightarrow 0$;

b) The limit of A_{mn} depends on how $m, n \longrightarrow \infty$. Furthermore for any $s > 2\pi$ (including ∞), there is a strictly increasing function $M: \mathbb{N} \mapsto \mathbb{N}$ such that

$$\lim_{n\to\infty} A_{M(n),n} = s. \tag{22}$$

Note that the area of the cylinder is 2π.

Remark 24. See http://www.cut-the-knot.org/Outline/Calculus/SchwarzLantern.shtml for a visualization of the construction.

Remark 25. (From (Lord)) In 1868 J. A. Serret[3] suggested the "obvious" generalization of the natural method of finding arc length to calculation of surface area:

2. Tibor Rado, *What is the Area of a Surface?*, The American Mathematical Monthly, Vol. 50, No. 3, Mar., 1943, pp. 139 - 141.

3. of Frenet-Serret frame in Differential Geometry.

> "Given a portion of a curved surface bounded by a curve C, we call the area of this surface the limit S towards which the area of an inscribed polyhedral surface tends, where the inscribed polyhedral surface is formed by triangular faces and is bounded by the polygonal curve G, which limits the curve C"
>
> "One must show that the limit S exists and that it is independent of the way in which the faces of the inscribed surface decreases."

The problem with this approach was first realized by H. A. Schwarz[4], who wrote to Italian mathematician Gennochi about this in 1880. Later in 1882 Gennochi's student Peano annouced the same result in a course he taught. Around the same time Schwarz wrote to Hermite about his example. Hermite published Schwarz's letter in his course notes, which was published later than that of Peano's. Consequently there are disputes about priority.

4. Gesammelte Mathematische Abhandlungen, Vol. 2, p. 309. Berlin, Julius Springer, 1890.