Math 348 Differential Geometry of Curves and Surfaces

Lecture 3 Curves in Calculus

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Please do not hesitate to interrupt me if you have a question. .

How to Define a Curve Mathematically?

Mathematical Representations of Curves

- **Level Set.**
  - Motivation: Analytic geometry, graphs.
  - 2D. $f(x, y) = 0$.
  - 3D. $f_1(x, y, z) = f_2(x, y, z) = 0$.
  - nD?
- **Parametrization.**
  - Motivation: Mechanics.
  - 2D. $(x(t), y(t))$.
  - 3D. $(x(t), y(t), z(t))$.
  - nD?
- **What About Graphs in 2D?**
  - $y = f(x)$ is both.

Example

A straight line in $\mathbb{R}^3$ represented

1. through level sets.
2. through parametrization.

Pros and Cons

- **Level Set Representations.**
  - Includes ”non-curves”. (Whitney's Theorem[1].)
  - In-efficient for theoretical study.
  - Efficient computational study (Level Set Method[2]).
- **Parametrization.**
  - Includes ”non-curves”. (Space-filling curves, etc. Example: Hilbert curve[3].)
  - In-efficient for computational study.
  - Efficient for theoretical study.
- **Our Choice.** Parametrization.

[1] Wiki Page
[2] Wiki Page, One of the inventor's explanation.
[3] Wiki Page

Regular Curves

- **Parametrized Curve.** A parametrized curve in $\mathbb{R}^n$ is a map $\gamma : (\alpha, \beta) \mapsto \mathbb{R}^n$, for some α, β with $-\infty \leqslant \alpha < \beta \leqslant \infty$.
- **Regular Curve.** A regular curve is a parametrized curve further satisfying
    1. $\gamma \in C^\infty$, that is γ is infinitely differentiable.
    2. $\dot{\gamma}(t) \neq 0$ at every $t \in (\alpha, \beta)$.
- **Notable Things.**
    1. $(\alpha, \beta) = \{t \in \mathbb{R} \mid \alpha < t < \beta\}$.
    2. Trace: The collection of points on a curve, i.e. $\gamma((\alpha, \beta))$.

Example

$\gamma(t) = (\cos t, \sin t), t \in (0, 4\pi)$ is a regular curve, whose trace is the unit circle in the plane.

Note that $\gamma_1(t) = (\cos 2t, \sin 2t), t \in (-\infty, \infty)$ is a different regular curve, but with the same trace.

In the remaining of the course, "curve" = "regular curve".

Example

- $\gamma(t) = (t, t^2),\ t \in (0, \infty)$.
- $\gamma(t) = (t^2, t^4),\ t \in (0, \infty)$.

Example

1. $\gamma(t) = (a + r\cos t, b + r\sin t), t \in (-\infty, \infty)$;
2. $\gamma(t) = (x_0 + tu_0, y_0 + tv_0, z_0 + tw_0), t \in (0, \infty)$;
3. $\gamma(t) = (t - \sin t, 1 - \cos t), t \in (0, 2\pi)$;
4. $\gamma(t) = (3\cos t, 4\sin t, 5t), t \in (-\infty, \infty)$;
5. Viviani's Curve (MathWorld Page).
   $\gamma(t) = (R\cos^2 t, R\cos t\sin t, R\sin t), \qquad t \in (0, 2\pi)$.

Differentiation

- Differential! Geometry.
- First Order Derivative.
  - $\gamma(t) : (x(t), y(t), z(t)), t \in (\alpha, \beta)$.
  - $\dot{\gamma}(t_0) = (\dot{x}(t_0), \dot{y}(t_0), \dot{z}(t_0))$.
  - Mechanics: $\gamma(t)$ is trajectory, t is time $\Longrightarrow \dot{\gamma}(t_0)$ is velocity at time t_0.
  - Geometry: $\dot{\gamma}(t_0)$ is a vector that is tangent to the curve.[4]
- Higher Order Derivatives.
  - Mathematics: $\ddot{\gamma}(t_0 = (\ddot{x}(t_0), \ddot{y}(t_0), \ddot{z}(t_0))$, etc..
  - Mechanics: Acceleration, etc..
  - Geometry: ???

[4] The vector may not be tangent to the trace of the curve...

How to Measure Along a Curve?

Arc Length

The Arc Length Formula

$$\boxed{L = \int_a^b \|\gamma'(t)\| \,\mathrm{d}t.}$$

- **Motivation.**

  Curve $\leftrightarrow$ particle trajectory $\implies$ Arc length $\leftrightarrow$ distance travelled.

- **Mathematical Definition.**
  - Idea. Approximate by polygonal curves.
  - $L = \sup\{$Length of all possible polygonal approximations.$\}$
- **Proof.**
  - Why is a proof necessary?
  - Proof itself optional. See lecture notes.

Calculation of Arc Length

Example

1. The circumference of the unit circle;
2. The helix $(\cos t, \sin t, t), t \in (0, 2\pi)$;
3. The cycloid[5] $(t - \sin t, 1 - \cos t), t \in (0, 2\pi)$.
4. The Limacon of Pascal [6] of Pascal $((1 + 2cost)cost, (1 + 2cost)sint)$.

- Arc length is independent of parametrization.

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[5] MathWorld Page
[6] Wiki Page

Arc Length Parametrization

The most convenient parametrization for the study of curves through calculus.

- **Definition.** An arc length parametrization of a curve γ is a parametrization $\gamma(s)$ such that $\|\dot{\gamma}(s)\| = 1$ at every s.
- **How to parametrize by arc length.** Let a curve γ be given.
    1. Calculate $\|\dot{\gamma}(t)\|$.
    2. Solve $\frac{\mathrm{d}S(t)}{\mathrm{d}t} = \|\dot{\gamma}(t)\|$. $s = S(t)$ is the new, arc length, parameter.
    3. Find the inverse function $t = T(s)$.
    4. Arc length parametrization is $\gamma(T(s)) = (x(T(s)), y(T(s)), z(T(s)))$.

Example

Parametrize $(cost, sint, t)$ by arc length.

What Can We Do With Naïve Application of Calculus?

Example

Let $\gamma(t)$ be a given curve. Then it is on a sphere centered at 0 if and only if $\dot{\gamma}(t) \cdot \gamma(t) = 0$ for every t.

Proof.

(Main idea)

1. $\frac{\mathrm{d}}{\mathrm{d}t} \|\gamma(t)\|^2 = 2\dot{\gamma}(t) \cdot \gamma(t)$.
2. The conclusion follows.

□

Straight Lines

Example

Let $\gamma(t)$ be a given curve in $\mathbb{R}^3$. Then it is part of a straight line passing the origin if and only if $\dot{\gamma}(t) \times \gamma(t) = 0$ for every t.

Proof.

(Main Idea)

- Only if. Let $\gamma(t)$ be a straight line. Then there is a constant vector b such that $\gamma(t) = f(t)b$ with $f(t)$ a scalar function. Conclusion follows.
- If.
    1. Let $b(t) = \gamma(t) / \|\gamma(t)\|$.
    2. Let $f(t) = \|\gamma(t)\|$.
    3. $\dot{\gamma}(t) \times \gamma(t) = f(t)\dot{b}(t) \times \gamma(t)$.
    4. $\dot{b}(t) = 0$. Conclusion follows.

□

Planar Curves

Example

Let $\gamma(t)$ be a given curve. Then it lies in a plane passing 0 if and only if $(\gamma(t) \times \dot{\gamma}(t)) \cdot \ddot{\gamma}(t) = 0$ for every t.

Proof.

(Main Idea)

- Only if. There is a constant nonzero vector b such that $b \cdot \gamma(t) = 0$ for all t.
- If.
  - Case 1. $\gamma(t) \times \dot{\gamma}(t) = 0$. Conclusion follows from previous example.
  - Case 2. $\gamma(t) \times \dot{\gamma}(t) \neq 0$.
    1. Let $b(t) = \gamma(t) \times \dot{\gamma}(t)$.
    2. Calculate $b(t) \times \dot{b}(t) = 0$.
    3. $b = b(t)/\left\|b(t)\right\|$ is a constant vector satisfying $b \cdot \gamma(t) = 0$ for all t.
    4. Conclusion follows.

□

Looking Back and Forward

Summary

- **Required Textbook Sections.** §1.1, §1.2.
- **Optional Textbook Sections.** §1.3–§1.5.

- **Definitions.**
  1. Regular Curve.

     A map $\gamma : (\alpha, \beta) \mapsto \mathbb{R}^n$, for some $-\infty \leqslant \alpha < \beta \leqslant \infty$, such that $\gamma \in C^\infty$, and $\dot{\gamma}(t) \neq 0$ for every t.
  2. Tangent Vector.

     The tangent vector of a curve $\gamma(t)$ at t_0 is $\dot{\gamma}(t_0)$.
- **Formulas.**
  1. Arc Length.

     $$L = \int_a^b \|\gamma'(t)\| \, \mathrm{d}t.$$
- **Procedures.**
  1. Arc Length Parametrization.

     1.1 Calculate $\|\dot{\gamma}(t)\|$.

     1.2 Solve $\frac{\mathrm{d}S(t)}{\mathrm{d}t} = \|\dot{\gamma}(t)\|$. $s = S(t)$ is the new, arc length, parameter.

     1.3 Find the inverse function $t = T(s)$.

     1.4 Arc length parametrization is $\gamma(T(s)) = (x(T(s)), y(T(s)), z(T(s)))$.

See You Thursday!

Surfaces in Calculus

- Mathematical representations of surfaces;
- Tangent planes;
- Measuring surface area.