# Lecture 3: Curves in Calculus

**Disclaimer.** As we have a textbook, this lecture note is for guidance and supplement only. It should not be relied on when preparing for exams.

In this lecture we set up the "play ground" for the mathematical study of curves in the 2-dimensional plane and the 3-dimensional space. In particular, we will discuss appropriate mathematical representation of curves, define their tangent vectors and arc lengths, and derive formulas for these quantities.

The required textbook sections are §1.1, §1.2. The optional textbook sections are §1.3, §1.4, §1.5.

The examples in this note are different from examples in the textbook. Please read the textbook carefully and try your hands on the exercises. During this please don't hesitate to contact me if you have any questions.

## Table of contents

# 1. Parametrization of Curves

The most convenient way (for application of calculus) to represent a spatial curve mathematically is to treat it as the trajectory of a moving particle. This leads to the so-called "parametrization" of the curves. There are infinitely many possible parametrizations of the same curve.

## 1.1. Mathematical representations of a curve

- LEVEL CURVES. Before the advent of calculus, a curve is usually defined through level sets:

  i. (in the plane) as level sets: $f(x,y)=0$;

  ii. (in the space) as intersection of surfaces (intersection of level sets):

  $$f_1(x,y,z)=0, \qquad f_2(x,y,z)=0. \tag{1}$$

  **Example 1.** A circle in $\mathbb{R}^2$ is represented as

  $$f(x,y)=(x-a)^2+(y-b)^2-r^2=0. \tag{2}$$

  A straight line in $\mathbb{R}^3$ is represented as

  $$a_1 x_1+a_2 x_2+a_3 x_3-a=0, \qquad b_1 x_1+b_2 x_2+b_3 x_3-b=0. \tag{3}$$

  The basic idea is to replace the study of the curve by the study of one or more simple surfaces.

- PARAMETRIZED CURVES. To apply calculus, the most convenient representation of a curve is through parametrization.[1]

  DEFINITION 2. (DEFINITION 1.1.1 IN TEXTBOOK) *A parametrized curve in* $\mathbb{R}^n$ *is a map*[2] $\gamma\colon(\alpha,\beta)\mapsto\mathbb{R}^n$, *for some* $\alpha,\beta$ *with* $-\infty\leqslant\alpha<\beta\leqslant\infty$.

  **Note.** Recall that $(a,b)$ means the set of all numbers $>a$ and $<b$, while $[a,b]$ means the set of all numbers $\geqslant a$ and $\leqslant b$.

  **Exercise 1.** Do you remember/Can you guess what $(a,b]$ and $[a,b)$ means?

  **Example 3.** A circle in $\mathbb{R}^2$ is represented as

  $$\gamma(t)=\begin{pmatrix} x(t)\\ y(t)\end{pmatrix}=\begin{pmatrix} a+r\cos t\\ b+r\sin t\end{pmatrix} \tag{4}$$

  for $t\in(\alpha,\beta)$ with $\alpha<0,\beta>2\pi$.

1. The reason for such transition is that, without calculus, there is no way to study a curve directly. Instead one has to look for simple functions that could "generate" the curve and turn the study of the curves to the study of those functions.

2. Another name for "function".

**Example 4.** A straightline in $\mathbb{R}^3$ is represented as

$$\gamma(t) = \begin{pmatrix} x(t) \\ x(t) \\ x(t) \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} + t \begin{pmatrix} u_0 \\ v_0 \\ w_0 \end{pmatrix} \tag{5}$$

for $t \in (-\infty, \infty)$.

**Example 5.** (CYCLOID) The trajectory of a unit circle rolling in the plane along a line.

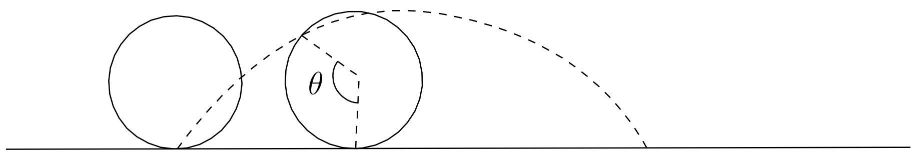


**Figure 1.** The cycloids

We take $t = \theta$ as the parameter. We have ⟨gmma⟩$(t) = (t - \sin t, 1 - \cos t)$ for $t \in [0, 2\pi]$.

**Exercise 2.** Write down a parametrized representation of a unit circle rolling outside (epicycloid) or inside (hypocycloid) another circle with radius $R$ centered at the origin (for the "inside" case, assume $R > 1$). Plot the following two cases: Outside with $R = 1$ (Cardioid); Inside with $R = 4$ (Astroid). [3] What happens if $R \longrightarrow \infty$?

**Example 6.** (VIVIANI'S CURVE) The intersection of a sphere $x^2 + y^2 + z^2 = R^2$ with $x^2 + y^2 = R x$. It can be parametrized by

$$\gamma(t) = (R \cos^2 t, R \cos t \sin t, R \sin t), \qquad t \in [0, 2\pi) \tag{6}$$

**Exercise 3.** Check Ex. 1.1.8 in the textbook for what it looks like. Compare (6) with the formula there. Can you find a different parametrization?

**Exercise 4.** Consider a plane curve given by a graph: $y = f(x)$. Is it a level curve or a parametrized curve?

**Exercise 5.** Try to draw the parametrized curve $\gamma(t) = (\cos t, \sin t, 2t)$, $t \in (\pi, 3\pi)$.

**Exercise 6.** Try to draw $\gamma(t) = (t^3, t^2)$ for $t \in (-\infty, \infty)$.

**Remark 7.** It is clear that neither level set nor parametrized representations are unique.

- There are infinitely many pairs of planes that intersect along the same straight line;
- There are infinitely many $\gamma(t): (\alpha, \beta) \mapsto \mathbb{R}^n$ such that their image set coincide.

We emphasize that this non-uniqueness is in fact a good thing as it allows us to choose the most convenient of them. We will see how this works later.

3. $\left(\left(\frac{1}{R}+1\right)\cos\left(\frac{t}{R}\right) - \frac{1}{R}\cos\left(\frac{t}{R}+t\right), \left(\frac{1}{R}+1\right)\sin\left(\frac{t}{R}\right) - \frac{1}{R}\sin\left(\frac{t}{R}+t\right)\right)$.

In 348 we will only consider smooth curves, that is parametrized curves $\gamma(t) = (x_1(t), x_2(t), \ldots, x_n(t))$, $t \in (\alpha, \beta)$, with each $x_n(t)$ infinitely differentiable for all $t \in (\alpha, \beta)$.

**Exercise 7.** Show through examples that the image of a smooth curve may not look smooth.

# 2. Tangent Vectors and Arc Length

When a curve is viewed as the trajectory of a moving particle, its velocities are represented mathematically as "tangent vectors", and the length of the trajectory is the "arc length".

## 2.1. From possible velocities to angent vectors

- FROM POSSIBLE VELOCITIES TO TANGENT VECTORS.
  - Let $\gamma(t) \in \mathbb{R}^n$ be the trajectory of a particle. Then its velocity at $t_0$ is given by
$$v(t_0) = \lim_{t \longrightarrow t_0} \frac{\gamma(t) - \gamma(t_0)}{t - t_0} = \frac{\mathrm{d}\gamma}{\mathrm{d}t}(t_0) = \dot{\gamma}(t_0). \tag{7}$$
  - For a parametrized curve $\gamma(t)$, we will view it as a mathematical model of a trajectory of a particle and define its velocity at $\gamma(t_0)$ to be one tangent vector at $\gamma(t_0)$.

DEFINITION 8. (TANGENT VECTOR) *Let $C$ be a curve in $\mathbb{R}^n$. Let $x_0 \in C$. Then $v \in \mathbb{R}^n$ is a tangent vector of $C$ at $x_0$ if and only if there is a parametrization $\gamma(t)$: $(\alpha, \beta) \mapsto \mathbb{R}^n$ of $C$ such that*

*i. there is $t_0 \in (\alpha, \beta)$ with $\gamma(t_0) = x_0$;*

*ii. $\dot{\gamma}(t_0) = v$.*

**Exercise 8.** Let $C$ be a curve in $\mathbb{R}^n$. Let $x_0 \in C$. Let $v$ be a tangent vector of $C$ at $x_0$.

i. Prove that $-v$ is also a tangent vector of $C$ at $x_0$;

ii. Prove that $2\,v$ is also a tangent vector of $C$ at $x_0$;

iii. Convince yourself that for arbitrary $c \in \mathbb{R}$, $c\,v$ is a tangent vector of $C$ at $x_0$;

iv. (optional) Prove that for arbitrary $c \in \mathbb{R}$, $c\,v$ is a tangent vector of $C$ at $x_0$.

The tangent vector of $\gamma(t)$ at $x_0 = \gamma(t_0)$: $\dot{\gamma}(t)$

- REGULAR CURVES.
  - A parametrized curve $\gamma(t)$: $(\alpha, \beta) \mapsto \mathbb{R}^n$ is regular if and only if it is smooth and $\dot{\gamma}(t) \neq 0$ for all $t \in (\alpha, \beta)$.

**Example 9.** The parametrized curve $\gamma(t) = \begin{pmatrix} t \\ t^2 \\ t^3 \end{pmatrix}$ is regular. To see this we check

i. $t, t^2, t^3$ are all smooth;

ii. $\dot{\gamma}(t) = \begin{pmatrix} 1 \\ 2t \\ 3t^2 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ no matter what $t$ is.

From now on "curve" means "regular curve".

## 2.2. Arc length of a parametrized curve

- THE ARC LENGTH FORMULA.
  - Let $\gamma(t) \colon (\alpha, \beta) \mapsto \mathbb{R}^n$ be a parametrized curve. Let $[a, b] \subset (\alpha, \beta)$. We try to obtain its arc length $L$ through the following: Pick $a = t_0 < \cdots < t_k = b$. Consider the straight line segments $\overline{\gamma(t_0)\gamma(t_1)}$, $\overline{\gamma(t_1)\gamma(t_2)}$, $\overline{\gamma(t_2)\gamma(t_3)}$, ... , $\overline{\gamma(t_{k-1})\gamma(t_k)}$. Then intuitively we have $L \geqslant$ the sum of the lengths of these line segments.
  - Inspired by the above, one defines

$$L := \sup_{a = t_0 < \cdots < t_k = b} \sum_{i=0}^{k-1} \|\gamma(t_i) - \gamma(t_{i-1})\| \tag{8}$$

    where we emphasize that the supreme is taken over all possible partitions $a = t_0 < \cdots < t_k = b$. In particular, $k \in \mathbb{N}$ is not fixed.
  - The formula.

$$\text{Arc length from } \gamma(a) \text{ to } \gamma(b) = \int_a^b \|\dot{\gamma}(t)\| \, \mathrm{d}t.$$

THEOREM 10. *Let $\gamma(t)$ be a smooth parametrized curve. Then its arc length from $\gamma(a)$ to $\gamma(b)$ is given by*

$$L = \int_a^b \|\dot{\gamma}(t)\| \, \mathrm{d}t. \tag{9}$$

**Proof.** (OPTIONAL; YOU MAY WANT TO READ THE BOX AFTER THE PROOF FIRST) We prove it in two steps.

1. $L \leqslant \int_a^b \|\dot{\gamma}(t)\| \, \mathrm{d}t$.

   By fundamental theorem of calculus, we have

$$\gamma(t_i) - \gamma(t_{i-1}) = \int_{t_{i-1}}^{t_i} \dot{\gamma}(t) \, \mathrm{d}t. \tag{10}$$

   Thus we have

$$\begin{aligned} \sum_{t=0}^{k-1} \|\gamma(t_i) - \gamma(t_{i-1})\| &= \sum_{t=0}^{k-1} \left\| \int_{t_{i-1}}^{t_i} \dot{\gamma}(t) \, \mathrm{d}t \right\| \\ &\leqslant \sum_{t=0}^{k-1} \int_{t_{i-1}}^{t_i} \|\dot{\gamma}(t)\| \, \mathrm{d}t \\ &= \int_a^b \|\dot{\gamma}(t)\| \, \mathrm{d}t. \end{aligned} \tag{11}$$

2. $L \geqslant \int_a^b \|\dot{\gamma}(t)\| \,\mathrm{d}t$.

   Let $k \in \mathbb{N}$ and define $t_i := a + \frac{i}{k}(b - a)$. Then we have $a = t_0 < t_1 < \cdots < t_k = b$.

   By fundamental theorem of calculus, we have

$$\gamma(t_i) - \gamma(t_{i-1}) = \int_{t_{i-1}}^{t_i} \dot{\gamma}(t)\,\mathrm{d}t = \dot{\gamma}(t_{i-1})\,(t_i - t_{i-1}) + R_i \tag{12}$$

where

$$R_i := \int_{t_{i-1}}^{t_i} [\dot{\gamma}(t) - \dot{\gamma}(t_{i-1})]\,\mathrm{d}t = \int_{t_{i-1}}^{t_i} \left[\int_{t_{i-1}}^{t} \ddot{\gamma}(s)\,\mathrm{d}s\right] \mathrm{d}t. \tag{13}$$

As $x(t)$ is a regular curve, there is $M > 0$ such that

$$\|\ddot{\gamma}(s)\| \leqslant M \qquad \text{for all } s \in [a, b]. \tag{14}$$

Consequently

$$\|R_i\| \leqslant \int_{t_{i-1}}^{t_i} \int_{t_{i-1}}^{t_i} M\,\mathrm{d}s\,\mathrm{d}t = M\,(t_i - t_{i-1})^2 = \frac{M\,(b-a)^2}{k^2}. \tag{15}$$

Thus we have

$$\begin{aligned} \sum_{i=0}^{k-1} \|\gamma(t_i) - \gamma(t_{i-1})\| &= \sum_{i=0}^{k-1} \|\dot{\gamma}(t_{i-1})\,(t_i - t_{i-1}) + R_i\| \\ &\geqslant \sum_{i=0}^{k-1} [\|\dot{\gamma}(t_{i-1})\,(t_i - t_{i-1})\| - \|R_i\|] \\ &\geqslant \sum_{i=0}^{k-1} \|\dot{\gamma}(t_{i-1})\|\,(t_i - t_{i-1}) - \frac{M\,(b-a)^2}{k^2}. \end{aligned} \tag{16}$$

On the other hand, we have

$$\begin{aligned} \int_{t_{i-1}}^{t_i} \|\dot{\gamma}(t)\|\,\mathrm{d}t &= \int_{t_{i-1}}^{t_i} [\|\dot{\gamma}(t)\| - \|\dot{\gamma}(t_{i-1})\|]\,\mathrm{d}t + \\ &\quad + \|\dot{\gamma}(t_{i-1})\|\,(t_i - t_{i-1}) \\ &\leqslant \|\dot{\gamma}(t_{i-1})\|\,(t_i - t_{i-1}) + \int_{t_{i-1}}^{t_i} \|\dot{\gamma}(t) - \dot{\gamma}(t_{i-1})\|\,\mathrm{d}t \\ &= \|\dot{\gamma}(t_{i-1})\|\,(t_i - t_{i-1}) + \\ &\quad + \int_{t_{i-1}}^{t_i} \left\| \int_{t_{i-1}}^{t} \|\ddot{\gamma}(s)\|\mathrm{d}s \right\| \mathrm{d}t \\ &\leqslant \|\dot{\gamma}(t_{i-1})\|\,(t_i - t_{i-1}) + M\,(t_i - t_{i-1})^2 \\ &= \|\dot{\gamma}(t_{i-1})\|\,(t_i - t_{i-1}) + \frac{M\,(b-a)^2}{k^2}. \end{aligned} \tag{17}$$

From this we conclude

$$\sum_{i=0}^{k-1} \|\gamma(t_i) - \gamma(t_{i-1})\| - \int_a^b \|\dot{\gamma}(t)\|\,\mathrm{d}t \geqslant -\frac{2\,M\,(b-a)^2}{k^2} \longrightarrow 0 \tag{18}$$

as $k \longrightarrow \infty$. This implies

$$L := \sup_{a=t_0<t_1<\cdots<t_k=b} \sum_{i=0}^{k-1} \|\gamma(t_i) - \gamma(t_{i-1})\| \geqslant \int_a^b \|\dot{\gamma}(t)\|\,\mathrm{d}t \tag{19}$$

and ends the proof. □

***Technical Aside***

In the above proof we have used the following results from calculus.

– Supreme. Let $A$ be a collection (set) of numbers. Then its supreme $\sup A$ is the smallest number that is greater than or equal to $A$. As a consequence,

- to prove that $\sup A \leqslant b$, all we need to do is to show that for every $a \in A$, there holds $a \leqslant b$;
- to prove that $\sup A \geqslant b$, all we need to do is to find one particular sequence of $a_n \in A$ such that $\lim_{n\rightarrow\infty} (a_n - b) \geqslant 0$.

**Exercise 9.** What do we need to do to prove $\sup A > b$ or $\sup A < b$?

– Triangle inequality.

- Classical triangle inequality:

$$\|x\| + \|y\| \geqslant \|x + y\|. \tag{20}$$

- Variant:

$$|\|x\| - \|y\|| \leqslant \|x - y\|. \tag{21}$$

- Generalization:

$$\|x_1\| + \cdots + \|x_k\| \geqslant \|x_1 + \cdots + x_k\|. \tag{22}$$

- Integral: Together with the definite of Riemann integrals, (22) yields the following inequality for vector functions:

$$\int_a^b \|x(t)\|\,\mathrm{d}t \geqslant \left\|\int_a^b x(t)\,\mathrm{d}t\right\|. \tag{23}$$

- Examples.

**Example 11.** Calculate the circumference of the unit circle $x^2 + y^2 = 1$.

**Solution.**

- Method 1. We calculate the curve length $l$ of the graph $y=\sqrt{1-x^2}, -1\leqslant x\leqslant 1$. Then the circumference is $L=2\,l$.

$$\begin{aligned} l &= \int_{-1}^{1}\sqrt{1+[(\sqrt{1-x^2})']^2}\,\mathrm{d}x \\ &= \int_{-1}^{1}\frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x \\ &\overset{x=\sin t}{=\!=} \int_{-\pi/2}^{\pi/2}\mathrm{d}t=\pi. \end{aligned} \tag{24}$$

  So the circumference is L=$2\,\pi$.

- Method 2. We parametrize $x(t)=\cos t$, $y(t)=\sin t$,$0\leqslant t<2\,\pi$. Then

$$L=\int_0^{2\pi}\sqrt{x'(t)^2+y'(t)^2}\,\mathrm{d}t=2\,\pi. \tag{25}$$

**Example 12.** Calculate the arc length of the space curve

$$x=\cos t,\, y=\sin t,\, z=t \tag{26}$$

for $t$ from 0 to $2\,\pi$.

**Solution.** We have

$$\begin{aligned} L &= \int_0^{2\pi}\sqrt{x'(t)^2+y'(t)^2+z'(t)^2}\,\mathrm{d}t \\ &= \int_0^{2\pi}\sqrt{2}\,\mathrm{d}t=2\,\sqrt{2}\,\pi. \end{aligned} \tag{27}$$

**Example 13.** Calculate the arc length of the cycloid $(t-\sin t, 1-\cos t)$ from $t=0$ to $2\,\pi$.

**Solution.** We have

$$\begin{aligned} L &= \int_0^{2\pi}\sqrt{(1-\cos t)^2+(\sin t)^2}\,\mathrm{d}t \\ &= \int_0^{2\pi}\sqrt{2-2\cos t}\,\mathrm{d}t \\ &= \int_0^{2\pi}\sqrt{2-2\left(1-2\left(\sin\frac{t}{2}\right)^2\right)}\,\mathrm{d}t \\ &= 2\int_0^{2\pi}\left|\sin\left(\frac{t}{2}\right)\right|\mathrm{d}t \\ &= 8. \end{aligned} \tag{28}$$

**Example 14.** Calculate the arc length of the Limacon of Pascal $((1+2\cos t)\cos t$, $(1+2\cos t)\sin t)$, $t\in(0,2\,\pi)$.

**Solution.** We have $\gamma(t) = (\cos t + 2\cos^2 t, \sin t + \sin 2t)$.

$$L = \int_0^{2\pi} \sqrt{(-\sin t - 2\sin 2t)^2 + (\cos t + 2\cos 2t)^2}\, dt$$
$$= \int_0^{2\pi} \sqrt{5 + 4\cos t}\, dt. \quad (29)$$

It turns out that this integral cannot be calculated explicitly.

**Exercise 10.** Calculate the arc length of $x = \cos^3 t$, $y = \sin^3 t$, $t \in [0, 2\pi)$.

**Exercise 11.** (Optional) It is clear that arc length should be independent of parametrization. Prove this.

## 3. Arc Length Parametrization

Among the infinitely many possible parametrizations of the same curve, one particular parametrization, called "arc length parametrization" stands out as the most convenient due to its ability to simplify calculations.

- Motivation. For a general parametrized curve $\gamma(t)$: $(\alpha, \beta) \mapsto \mathbb{R}^n$, in general we have $\|\dot{\gamma}(t)\|$ varying with $t$. As we will see later, many calculations could be greatly simplified if $\|\dot{\gamma}(t)\| = 1$ for all $t \in (a, b)$. Such a parametrization is called the "arc length" parametrization of the curve.

  **Exercise 12.** In this case we have the arc length between $x(t_0)$ and $x(t_1)$ to be $t_1 - t_0$.

- Existence of arc length parametrization.

  Theorem 15. *Let $\gamma(t)$: $(\alpha, \beta) \mapsto \mathbb{R}^n$ be a regular curve. Then there is a strictly increasing function $T: (a, b) \mapsto (\alpha, \beta)$ such that the curve $\Gamma(s) := \gamma(T(s))$ is parametrized by its arc length.*

  Notation. *The convention is that, when the parametrization is arc length parametrization, we use $s$ as the variable. That is, when we write a curve as $x(s)$, we assume it is already parametrized by arc length.*

  **Proof.** (Optional) See Proposition 1.3.6 in the textbook. □

- Examples.

  **Example 16.** Consider the circle $(2\cos t, 2\sin t)$ in the plane. We have

  $$\|\dot{\gamma}(t)\| = \|(-2\sin t, 2\cos t)\| = 2. \quad (30)$$

  Thus the arc length parametrization is $(2\cos(s/2), 2\sin(s/2))$.

  **Example 17.** Consider the space curve

  $$x = \cos t, y = \sin t, z = t. \quad (31)$$

The arc length parametrization is $\big(\cos\big(s/\sqrt{2}\big), \sin\big(s/\sqrt{2}\big), s/\sqrt{2}\big)$.

## 4. Properties of curves from its derivatives

**Example 18.** $\|\gamma(t)\| = \text{constant}$ if and only if $\dot{\gamma}(t)\cdot\gamma(t) = 0$ for all $t$.

**Proof.** We have
$$\frac{\mathrm{d}}{\mathrm{d}t}\|\gamma(t)\|^2 = \frac{\mathrm{d}}{\mathrm{d}t}(\gamma(t)\cdot\gamma(t)) = 2\,\dot{\gamma}(t)\cdot\gamma(t). \tag{32}$$
The conclusion now follows. □

**Example 19.** Let $\gamma(t)$ be a curve in $\mathbb{R}^3$. Then the following two are equivalent.

a) There is a nonzero constant vector $b$ such that $\gamma(t) = f(t)\,b$ for some scalar function $f(t) \neq 0$.

b) $\dot{\gamma}(t)\times\gamma(t) = 0$ for all $t$.

**Proof.**

- a)$\Longrightarrow$b).
  We have
$$\begin{aligned}\dot{\gamma}(t)\times\gamma(t) &= \big(\dot{f}(t)\,b\times f(t)\,b\big)\\ &= \dot{f}(t)\,f(t)\,(b\times b) = 0.\end{aligned} \tag{33}$$

- b)$\Longrightarrow$a).
  Let $b(t) := \frac{\gamma(t)}{f(t)}$ with $f(t) := \|\gamma(t)\|$. Then we have
$$\begin{aligned}0 &= \dot{\gamma}(t)\times\gamma(t)\\ &= \frac{\mathrm{d}}{\mathrm{d}t}(f(t)\,b(t))\times(f(t)\,b(t))\\ &= \big(\dot{f}(t)\,b(t) + f(t)\,\dot{b}(t)\big)\times(f(t)\,b(t))\\ &= \big(\dot{f}(t)\,b(t)\big)\times(f(t)\,b(t)) + \big(f(t)\,\dot{b}(t)\big)\times(f(t)\,b(t))\\ &= \dot{f}(t)\,f(t)\,(b(t)\times b(t)) + f^2(t)\,\big(\dot{b}(t)\times b(t)\big)\\ &= f^2(t)\,\big(\dot{b}(t)\times b(t)\big).\end{aligned} \tag{34}$$
  As $f(t)\neq 0$, we have $\dot{b}(t)\times b(t) = 0$, or $\dot{b}(t)$ is parallel to $b(t)$. On the other hand, as $\|b(t)\| = \left\|\frac{\gamma(t)}{\|\gamma(t)\|}\right\| = \frac{\|\gamma(t)\|}{\|\gamma(t)\|} = 1$, by Example 18 there holds $\dot{b}(t)\cdot b(t) = 0$ or $\dot{b}(t)\perp b(t)$. Consequently $\dot{b}(t) = 0$, or $b(t) = b$ is a constant vector. □

**Exercise 13.** What happens if we drop the assumption $f(t)\neq 0$?

**Example 20.** Let $\gamma(t)$ be a curve in $\mathbb{R}^3$. Then the following are equivalent.

a) There is a nonzero constant vector $b$ such that $b\perp\gamma(t)$ for all $t$;

b) $(\gamma(t)\times\dot{\gamma}(t))\cdot\ddot{\gamma}(t) = 0$ for all $t$.

**Proof.**

- a)$\Longrightarrow$b).

  We have

  $$b\cdot\dot{\gamma}(t)=\frac{\mathrm{d}}{\mathrm{d}t}(b\cdot\gamma(t))=0, \qquad b\cdot\ddot{\gamma}(t)=\frac{\mathrm{d}}{\mathrm{d}t}(b\cdot\dot{\gamma}(t))=0. \tag{35}$$

  Thus if we set the $3\times 3$ matrix $\Gamma=\left(\begin{smallmatrix}\gamma & \dot{\gamma} & \ddot{\gamma}\end{smallmatrix}\right)$, then $\Gamma\, b=0$. Conseqeuently, recalling $b\neq 0$,

  $$(\gamma\times\dot{\gamma})\cdot\ddot{\gamma}=\det\Gamma=0. \tag{36}$$

- b)$\Longrightarrow$a).

  - Case 1. $\gamma(t)\times\dot{\gamma}(t)=0$ for all $t$. Then by Example 19 $\gamma(t)=f(t)\,a$ for some constant vector $a$. Take $b\perp a$ and the conclusion follows.

  - Case 2. $\gamma(t)\times\dot{\gamma}(t)\neq 0$ for any $t$.

    Let $a(t):=\gamma(t)\times\dot{\gamma}(t)$. As $(\gamma\times\dot{\gamma})\cdot\ddot{\gamma}=0$, $\ddot{\gamma}$ is contained in the plane spanned by $\gamma$ and $\dot{\gamma}$. Consequently $\gamma\times\ddot{\gamma}$ is perpendicular to the same plane, and

    $$\begin{aligned} a(t)\times\dot{a}(t) &= (\gamma\times\dot{\gamma})\times(\gamma\times\ddot{\gamma}) \\ &= 0. \end{aligned} \tag{37}$$

    Now let $b(t):=\frac{a(t)}{\|a(t)\|}$.

    **Exercise 14.** Prove that $b(t)\times\dot{b}(t)=0$.

    By similar argument as in Example 19, we conclude that $\dot{b}(t)=0$, that is $b$ is a constant vector. Now the conclusion follows from $b\cdot\gamma(t)=0$ for all $t$. $\square$