Math 348 Differential Geometry of Curves and Surfaces

Lecture 2 Review of Pre-requisites

Xinwei Yu

Sept. 7, 2017

CAB 527, xinwei2@ualberta.ca
Department of Mathematical & Statistical Sciences
University of Alberta

Table of contents

Please do not hesitate to interrupt me if you have a question.

Pre-requisites to Classical Differential Geometry

Classical Differential Geometry

- **Objects to Study.**
  - Curves in $\mathbb{R}^2$ and $\mathbb{R}^3$;
  - Surfaces in $\mathbb{R}^3$.
- **Ultimate Goal.** Understand curves and surfaces through quantities that can be calculated. Questions to answer:
  - When are two curves identical[1]?
  - When are two surfaces identical?
  - When is a curve contained in a certain surface?
  - etc.
- **Idea.**
  - Calculate using calculus.
- **Procedure.**
  1. Represent curves and surfaces by functions;
  2. Study these functions by differentiation and integration;
  3. Interpret the results geometrically.

[1] What do we mean by "identical"?

Pre-requisites

- Multivariable Vector-valued Functions.
  - Operations in the Euclidean space $\mathbb{R}^n$;
  - Functions between two Euclidean spaces $\mathbb{R}^m$ and $\mathbb{R}^n$;
  - Geometric meanings of derivatives of these functions.
- Linear Algebra.
  - Efficient representations of $\mathbb{R}^n$.
  - Linear transformations and their matrix representations;
  - Properties of matrices and how to manipulate them.
- Differential Equations[2].
  - Solving simple ODEs;
  - Basic theory of differential equations.

[2] Geometry by algebra leads to algebraic equations; Geometry by calculus leads to differential equations.

Calculus of Multivariable Vector-valued Functions

The Euclidean Space $\mathbb{R}^n$

- nD Euclidean Space.
  - Represented by n-tuples of real numbers (coordinates) $(x_1, ..., x_n)$.
  - Original motivation: The ”ambient” space where particles live in.[3]
- Two Interpretations of $\mathbb{R}^n$.
  - Space for locations. $(x_1, ..., x_n)$ is a ”point”.
  - Space for velocities. $(v_1, ..., v_n)$ is a ”vector”.
  - Imagine at each point x in the location space $\mathbb{R}^n$, we could ”overlay” a velocity space $\mathbb{R}^n$.
- Understanding this distinction is crucial in DG.

[3] Their trajectories are curves; They may be restricted to surfaces.

$\mathbb{R}^n$ as a "Velocity" Space

- **Vectors.** Its members are called "vectors". The following are basic operations on vectors.
- **Addition/Subtraction/Scalar Multiplication.**
$$u \pm v = (u_1 \pm v_1, \ldots, u_n \pm v_n; \qquad au = (au_1, \ldots, au_n).$$
- **Norm.**
$$\|v\| = \sqrt{v_1^2 + \cdots + v_n^2}.$$
- **Inner Product.**
$$u \cdot v = u_1 v_1 + \cdots + u_n v_n.$$
- **Cross Product.** Let $u, v \in \mathbb{R}^3$.
$$u \times v = (u_2 v_3 - u_3 v_2, u_3 v_1 - u_1 v_3, u_1 v_2 - u_2 v_1).$$

Geometric Meanings of Vector Operations

- Addition/Subtraction/Scalar Multiplication.
- Norm.
- Inner Product.
$$\cos\theta = \frac{u \cdot v}{\|u\| \, \|v\|}.$$
- Cross Product.
$$u \times v \perp u, \qquad u \times v \perp v.$$
$$\sin\theta = \frac{\|u \times v\|}{\|u\| \, \|v\|}.$$
  $u \times v$ is a vector that is perpendicular to both u and v, its direction determined by the ”right-hand-rule”, with norm $\|u\| \, \|v\| \sin\theta$ where θ is the angle between u and v.

$\mathbb{R}^n$ as a "Location" Space: Measurements

- Distance.

$$d(x, y) = \sqrt{(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2}.$$

- Area.
  - Area is uniquely defined. If we agree on
    1. The unit square has area 1.
    2. The area of a disjoint union is the sum of the areas.
    3. Rigid movements does not change area
  - Area of triangles in $\mathbb{R}^3$.

    The area of the triangle

$$(0, 0, 0) - (x_1, x_2, x_3) - (y_1, y_2, y_3) - (0, 0, 0)$$

    is $\frac{1}{2}\|x \times y\|$.
- Volume.
  - Volume is also uniquely defined.
  - Volume of parallelopiped: $V = |(x \times y) \cdot z|$
- Moving with Velocity.

Functions

- **Notation.**

$$f : \mathbb{R}^m \mapsto \mathbb{R}^n.$$

- **Linear Transformations**[4]**.**
    - Definition: For all $a, b \in \mathbb{R}$,

    $$f(ax + by) = af(x) + bf(y)$$

    - NO.1 Property: Can be represented by matrices.

    $$f(x) = Ax.$$

- **Bilinear Forms**[5]**.** $f : \mathbb{R}^n \times \mathbb{R}^n \mapsto \mathbb{R}.$
    - Definition: For all $a, b \in \mathbb{R}$ and $u, v, w \in \mathbb{R}^n$,

    $$B(au+bw, v) = aB(u, v)+bB(w, v); \quad B(u, aw+bv) = aB(u, w)+bB(u, v).$$

    - NO.1 Property: Can be represented by matrices.

    $$B(u, v) = u \cdot Av.$$

[4] The most important function.
[5] The second most important function.

Differentiation

Differentiate $f : \mathbb{R}^m \mapsto \mathbb{R}^n$ at a point $x_0 \in \mathbb{R}^m$

- **Recall definition.**

  $Df(x_0)$ `is a linear transformation!`

- $Df(x_0)$.
  - A linear transformation from $\mathbb{R}^m$ to $\mathbb{R}^n$;
  - Not the original ”location” $\mathbb{R}^m$ and $\mathbb{R}^n$, but a pair of ”velocity” $\mathbb{R}^m$ and $\mathbb{R}^n$, attached to the two locations spaces at x_0 and $f(x_0$, respectively;
  - The ”Jacobian matrix” is its matrix representation.
  - Directional derivative:

$$\frac{\partial f}{\partial v}(x) = Df(x)(v).$$

More on Differentiation

- **Chain Rule.**

$$\frac{\partial (g \circ f)_i}{\partial x_j} = \sum_{k=1}^{n} \frac{\partial g_i}{\partial y_k}(f(x))\frac{\partial f_k}{\partial x_j}(x).$$

- **Second Order Derivative.**
  - A bilinear form.
  - Represented by the Hessian matrix.
- **Taylor Expansion.**

$$f(x+v) = f(x) + Df(x)(v) + \frac{1}{2}D^2 f(x)(v,v) + R.$$

Example

Let $f(x,y) = e^{xy}$. Its Taylor expansion at $(0,0)$ to second order is

$$f(u,v) = 1 + uv + R.$$

Linear Algebra

Linear (In)dependence

- **Representation of the velocity space.**

  Through a basis $v_1, \dots, v_k$: $v = a_1 v_1 + \cdots + a_k v_k$.

- **Represent the whole space.**

  $k \geqslant n$.

- **No redundancy.** :

  Each v_i is "independent" from all other v_j's.

- **Linear dependence/independence.**
  - Linear dependence of $v_1, \dots, v_k$: There are $a_1, \dots, a_k \in \mathbb{R}$, not all zero, such that

    $$a_1 v_1 + \cdots + a_k v_k = 0.$$

  - Let $v_1, \dots, v_k$ be linearly dependent. Then there is a v_i that is a linear combination of the other v_j's.
  - Linearly independent = Not linearly dependent.
  - Linear independence of $v_1, \dots, v_k \implies k \leqslant n$.

Determinant

- **The unique function on $n \times n$ matrices such that**
  1. Takes value 1 for the identity matrix;
  2. Changes sign when two columns are switched;
  3. Is a linear function for each column.
- **Formulas** for 2×2 and 3×3 matrices.
- **Geometric meaning.** Volume of parallelopiped.
- **Relation to linear (in)dependence.**
  Let $v_1, \dots, v_n \in \mathbb{R}^n$. Let V be the matrix with $v_1, \dots, v_n$ as columns. Then the vectors are linearly dependent if and only if $\det V = 0$.

Eigenvalues and Eigenvectors

`Most properties of an` $n \times n$ `matrix` A `are revealed through its eigenvalues.`
`Most of the remaining properties can be studied through also looking at its eigenvectors.`

- **Calculation of eigenvalues.**
  Solve $\det \lambda I - A = 0$.
- **Eigenvectors.** There are one or more eigenvectors associated with each eigenvalue. To calculate, solve

$$(\lambda I - A)v = 0$$

  where λ is an eigenvalue.
- **Diagonalization.** If there are n linearly independent eigenvectors $v_1, \ldots, v_n$, then $A = V\Lambda V^{-1}$. V: Matrix with columns v_i. Λ: Diagonal matrix of the eigenvalues.

Differential Equations

Example

The solution to

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(t), \qquad x(t_0) = x_0$$

is $x(t) = x_0 + \int_{t_0}^{t} f(s)\mathrm{d}s$.

Example

The solution to

$$\frac{\mathrm{d}x}{\mathrm{d}t}(t) + cx(t) = f(t), \qquad x(0) = x_0$$

is

$$x(t) = e^{-ct}x_0 + \int_0^t e^{c(s-t)}f(s)\mathrm{d}s.$$

Some ODE Theory

- ODE System.

$$
\begin{aligned}
\frac{\mathrm{d}x_1}{\mathrm{d}t} &= f_1(x_1, \ldots, x_n), \qquad x_1(0) = x_{01} \\
\vdots \quad & \vdots \quad \vdots \\
\frac{\mathrm{d}x_n}{\mathrm{d}t} &= f_n(x_1, \ldots, x_n), \qquad x_n(0) = x_{01}
\end{aligned}
$$

- Existence and Uniquenss.

  The solution exists and is unique if f is differentiable with continuous derivatives.

- Explicit solution when f is linear.

See You Tuesday!

Curves in Calculus

- Mathematical representations of curves;
- Tangent vectors;
- Measuring arc length for curves;
- Arc length parametrization.