Solutions to Homework 9

(Total 20 pts; Due Dec. 8 12pm)

Question 1. (5 pts) *Use the Gauss-Bonnet Theorem to prove that circles on the unit sphere that are not big circles are not geodesics.*

Proof. Assume the contrary. Let $\mathcal{C}$ be a non-big circle on the unit sphere that is a geodesic. Then it divides the unit sphere into two regions Ω_N and Ω_S with different areas. Without loss of generality assume $|\Omega_N| > 2\pi$. Now by Gauss-Bonnet we have

$$|\Omega_N| = \int_{\Omega_N} K + \int_{\mathcal{C}} \kappa_g = 2\pi. \tag{1}$$

Contradiction. □

Question 2. (5 pts) *Let $S_A, \ldots, S_Z$ be compact surfaces that look like $A, B, \ldots, Z$ respectively. How many values do the 26 integrals of Gaussian curvatures $\int_{S_A} K, \ldots, \int_{S_Z} K$ take?*

Solution. We classify

- No "hole": $S_C, S_E, S_F, S_G, S_I, S_J, S_K, S_L, S_M, S_N, S_S, S_T, S_U, S_V, S_W, S_X, S_Y, S_Z$. For them $\int K = 4\pi$;
- One "hole": $S_A, S_D, S_O, S_P, S_Q, S_R$. For them $\int K = 0$;
- Two "holes": S_B. For them $\int K = -4\pi$.

Thus they take three values.

Question 3. *Prove that geodesic curvature is invariant under isometries.*

Proof. Let $\gamma(s) = \sigma(u(s), v(s))$. We show that κ_g can be calculated using $\mathbb{E}, \mathbb{F}, \mathbb{G}$ and u, v only.

We have

$$T(s) = \dot{u}\,\sigma_u + \dot{v}\,\sigma_v, \tag{2}$$

and

$$\begin{aligned}\dot{T} &= \ddot{u}\,\sigma_u + \ddot{v}\,\sigma_v + \sigma_{uu}\,\dot{u}^2 + 2\,\sigma_{uv}\,\dot{u}\,\dot{v} + \sigma_{vv}\,\dot{v}^2 \\ &= \ddot{u}\,\sigma_u + \ddot{v}\,\sigma_v \\ &\quad + [\Gamma_{11}^1\,\sigma_u + \Gamma_{11}^2\,\sigma_v + \mathbb{L}\,N_S]\,\dot{u}^2 \\ &\quad + 2\,[\Gamma_{12}^1\,\sigma_u + \Gamma_{12}^2\,\sigma_v + \mathbb{M}\,N_S]\,\dot{u}\,\dot{v} \\ &\quad + [\Gamma_{22}^1\,\sigma_u + \Gamma_{22}^2\,\sigma_v + \mathbb{N}\,N_S]\,\dot{v}^2 \\ &= [\ddot{u} + \Gamma_{11}^1\,\dot{u}^2 + 2\,\Gamma_{12}^1\,\dot{u}\,\dot{v} + \Gamma_{22}^1\,\dot{v}^2]\,\sigma_u \\ &\quad + [\ddot{v} + \Gamma_{11}^2\,\dot{u}^2 + 2\,\Gamma_{12}^2\,\dot{u}\,\dot{v} + \Gamma_{22}^2\,\dot{v}^2]\,\sigma_v \\ &\quad + [\mathbb{L}\,\dot{u}^2 + 2\,\mathbb{M}\,\dot{u}\,\dot{v} + \mathbb{N}\,\dot{v}^2]\,N_S. \end{aligned} \tag{3}$$

Now recall that

$$\dot{T} = \kappa_n\,N_S + \kappa_g\,(N_S \times T). \tag{4}$$

We have

$$\kappa_g = \dot{T} \cdot (N_S \times T) = \left(T \times \dot{T}\right) \cdot N_S. \tag{5}$$

We see that κ_g can be represented by u, v and the Γ_{ij}^k's, and the conclusion follows. □