# Solutions to Homework 8

(Total 20 pts; Due Dec. 1 12pm)

Question 1. (5 pts) *Let $S_1$ be a surface patch parametrized by $\sigma_1(u, v)$. Let $S_2$ be the surface patch parametrized by $\sigma_2(u,v) = 2\,\sigma_1(u,v)$. Prove that the following relation holds for the Gaussian curvatures: $K_2(u,v) = \frac{1}{4} K_1(u,v)$.*

**Proof.** We have $\sigma_{2u} = 2\,\sigma_{1u}, \sigma_{2v} = 2\,\sigma_{1v}$. Consequently

$$\mathbb{E}_2 = 4\,\mathbb{E}_1, \qquad \mathbb{F}_2 = 4\,\mathbb{F}_1, \qquad \mathbb{G}_2 = 4\,\mathbb{G}_1. \tag{1}$$

Next we notice that $N_2 = N_1$ and $\sigma_{2uu} = 2\,\sigma_{1uu}$, $\sigma_{2uv} = 2\,\sigma_{1uv}$, $\sigma_{2vv} = 2\,\sigma_{1vv}$ and therefore

$$\mathbb{L}_2 = 2\,\mathbb{L}_1, \qquad \mathbb{M}_2 = 2\,\mathbb{M}_1, \qquad \mathbb{N}_2 = 2\,\mathbb{N}_1. \tag{2}$$

Consequently

$$K_2 = \frac{\mathbb{L}_2\,\mathbb{N}_2 - \mathbb{M}_2^2}{\mathbb{E}_2\,\mathbb{G}_2 - \mathbb{F}_2^2} = \frac{1}{2}\,\frac{\mathbb{L}_1\,\mathbb{N}_1 - \mathbb{M}_1^2}{\mathbb{E}_1\,\mathbb{G}_1 - \mathbb{F}_1^2} = \frac{1}{4}\,K_1. \tag{3}$$

□

QUESTION 2. (5 PTS) *Let $S$ be a surface with first fundamental form $3\,\mathrm{d}u^2 + 4\,\mathrm{d}v^2$. Let $\mathbb{L}\,\mathrm{d}u^2 + 2\,\mathbb{M}\,\mathrm{d}u\,\mathrm{d}v + \mathbb{N}\,\mathrm{d}v^2$ be its second fundamental form. Prove that $\mathbb{L}_{vv} = \mathbb{N}_{uu}$.*

**Proof.** It is easy to check that $\Gamma_{ij}^k = 0$ for all $i, j, k$. The Codazzi-Mainradi equations become

$$\mathbb{L}_v - \mathbb{M}_u = 0, \qquad \mathbb{M}_v - \mathbb{N}_u = 0. \tag{4}$$

Thus $\mathbb{L}_{vv} = \mathbb{M}_{uv} = \mathbb{M}_{vu} = \mathbb{N}_{uu}$. □

QUESTION 3. (10 PTS) *Consider a surface with first fundamental form $v\,\mathrm{d}u^2 + u^2\,\mathrm{d}v^2$ (we assume $v > 0$)..*

a) (5 PTS) *Calculate $\Gamma_{ij}^k$.*

b) (5 PTS) *Can this surface have second fundamental form $u^{-1}\,\mathrm{d}u\,\mathrm{d}v$? Justify your claim.*

**Solution.**

a) We have $\mathbb{E} = v, \mathbb{F} = 0, \mathbb{G} = u^2, \mathbb{L} = 0, \mathbb{M} = u^{-1}, \mathbb{N} = 0,\ K = \frac{\mathbb{L}\,\mathbb{N} - \mathbb{M}^2}{\mathbb{E}\,\mathbb{G} - \mathbb{F}^2} = -\frac{1}{v\,u^4}$.

$$\sigma_{uu} = \Gamma_{11}^1 \sigma_u + \Gamma_{11}^2 \sigma_v \Longrightarrow \sigma_{uu} \cdot \sigma_u = v\,\Gamma_{11}^1, \qquad \sigma_{uu} \cdot \sigma_v = \Gamma_{11}^2 u^2. \tag{5}$$

From these we solve

$$\Gamma_{11}^1 = 0, \qquad \Gamma_{11}^2 = -\frac{1}{2\,u^2}. \tag{6}$$

Similarly we solve

$$\Gamma_{12}^1 = \frac{1}{2\,v}, \qquad \Gamma_{12}^2 = \frac{1}{u}, \qquad \Gamma_{22}^1 = -\frac{u}{v}, \qquad \Gamma_{22}^2 = 0. \tag{7}$$

**(To grader: it is OK to simply use the formulas in the textbook to calculate the Christoffel symbols.)**

b) Thus the Codazzi-Mainradi equations read

$$\frac{1}{u^2} = \frac{1}{u^2}, \tag{8}$$

$$0 = 0. \tag{9}$$

The Gauss equations become

$$-\frac{1}{u^4} = \frac{1}{4\,v\,u^2}, \tag{10}$$

$$0 = 0, \tag{11}$$

$$0 = 0, \tag{12}$$

$$-\frac{1}{v\,u^2} = -\frac{1}{2\,v^2}. \tag{13}$$

We see that such surface does not exist.