Solutions to Homework 7

(Total 20 pts; Due Nov. 24 12pm)

Question 1. (10 pts) *Consider the surface patch* $\sigma(u,v)=(u,v,3u^2+2v^2)$. *Calculate the Christoffel symbols* Γ_{ij}^k *at the point* $(1,0,3)$ *by solving the equations* $\sigma_{uu}=\Gamma_{11}^1\sigma_u+\Gamma_{11}^2\sigma_v+\mathbb{L}N$, *etc.*

Solution. We have, at $(1,0,3)=\sigma(1,0)$,

$$\sigma_u=(1,0,6u)=(1,0,6), \qquad \sigma_v=(0,1,4v)=(0,1,0), \tag{1}$$

$$N=\frac{\sigma_u\times\sigma_v}{\|\sigma_u\times\sigma_v\|}=\frac{(-6,0,1)}{\sqrt{37}}, \tag{2}$$

$$\sigma_{uu}=(0,0,6), \qquad \sigma_{uv}=(0,0,0), \qquad \sigma_{vv}=(0,0,4). \tag{3}$$

Now we solve

$$\begin{pmatrix}0\\0\\6\end{pmatrix}=\Gamma_{11}^1\begin{pmatrix}1\\0\\6\end{pmatrix}+\Gamma_{11}^2\begin{pmatrix}0\\1\\0\end{pmatrix}+\frac{\mathbb{L}}{\sqrt{37}}\begin{pmatrix}-6\\0\\1\end{pmatrix}. \tag{4}$$

Check the second component we see that $\Gamma_{11}^2=0$. (4) now becomes

$$\begin{pmatrix}0\\0\\6\end{pmatrix}=\Gamma_{11}^1\begin{pmatrix}1\\0\\6\end{pmatrix}+\frac{\mathbb{L}}{\sqrt{37}}\begin{pmatrix}-6\\0\\1\end{pmatrix}, \tag{5}$$

which becomes the linear system

$$\begin{aligned}\Gamma_{11}^{1} - \frac{6}{\sqrt{37}}\,\mathbb{L} &= 0, & (6)\\ 6\,\Gamma_{11}^{1} + \frac{1}{\sqrt{37}}\,\mathbb{L} &= 6. & (7)\end{aligned}$$

Solving this we see that $\Gamma_{11}^{1} = \frac{36}{37}$.

Next as $\sigma_{uv} = (0,0,0)$ clearly $\Gamma_{12}^{1} = \Gamma_{12}^{2} = 0$.

Finally we solve

$$\begin{pmatrix} 0 \\ 0 \\ 4 \end{pmatrix} = \Gamma_{22}^{1}\begin{pmatrix} 1 \\ 0 \\ 6 \end{pmatrix} + \Gamma_{22}^{2}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \frac{\mathbb{N}}{\sqrt{37}}\begin{pmatrix} -6 \\ 0 \\ 1 \end{pmatrix}. \tag{8}$$

By inspection we immediately conclude $\Gamma_{22}^{2} = 0$. The equation now becomes

$$\begin{pmatrix} 0 \\ 0 \\ 4 \end{pmatrix} = \Gamma_{22}^{1}\begin{pmatrix} 1 \\ 0 \\ 6 \end{pmatrix} + \frac{\mathbb{N}}{\sqrt{37}}\begin{pmatrix} -6 \\ 0 \\ 1 \end{pmatrix} \tag{9}$$

or equivalently

$$\begin{aligned}\Gamma_{22}^{1} - \frac{6}{\sqrt{37}}\,\mathbb{N} &= 0,\\ 6\,\Gamma_{22}^{1} + \frac{1}{\sqrt{37}}\,\mathbb{N} &= 4.\end{aligned}$$

From these we have

$$37\,\Gamma_{22}^{1} = 24 \Longrightarrow \Gamma_{22}^{1} = \frac{24}{37}. \tag{10}$$

Summarizing, we have

$$\Gamma_{11}^{1} = \frac{36}{37}, \Gamma_{11}^{2} = \Gamma_{12}^{1} = \Gamma_{12}^{2} = 0, \Gamma_{22}^{1} = \frac{24}{37}, \Gamma_{22}^{2} = 0. \tag{11}$$

QUESTION 2. (5 PTS) *Consider a surface with first fundamental form* $(1+36\,u^2)\,\mathrm{d}u^2 + 48\,u\,v\,\mathrm{d}u\,\mathrm{d}v + (1+16\,v^2)\,\mathrm{d}v^2$. *Calculate the Christoffel symbols* Γ_{ij}^k *at* $(u,v)=(1,0)$ *through the formulas* $\Gamma_{11}^1 = \frac{\mathbb{G}\,\mathbb{E}_u - 2\,\mathbb{F}\,\mathbb{F}_u + \mathbb{F}\,\mathbb{E}_v}{2\,(\mathbb{E}\,\mathbb{G} - \mathbb{F}^2)}$, *etc.*

Solution. We have

$$\mathbb{E} = 1 + 36\,u^2, \qquad \mathbb{F} = 24\,\mathrm{uv}, \qquad \mathbb{G} = 1 + 16\,v^2. \tag{12}$$

Now at $(1,0)$, $\mathbb{E}\,\mathbb{G} - \mathbb{F}^2 = 37$, and

$$\Gamma_{11}^1 = \frac{\mathbb{G}\,\mathbb{E}_u - 2\,\mathbb{F}\,\mathbb{F}_u + \mathbb{F}\,\mathbb{E}_v}{2\,(\mathbb{E}\,\mathbb{G} - \mathbb{F}^2)} = \frac{72}{74} = \frac{36}{37}, \tag{13}$$

$$\Gamma_{11}^2 = \frac{2\,\mathbb{E}\,\mathbb{F}_u - \mathbb{E}\,\mathbb{E}_v + \mathbb{F}\,\mathbb{E}_u}{2\,(\mathbb{E}\,\mathbb{G} - \mathbb{F}^2)} = 0, \tag{14}$$

$$\Gamma_{12}^1 = \frac{\mathbb{G}\,\mathbb{E}_v - \mathbb{F}\,\mathbb{G}_u}{2\,(\mathbb{E}\,\mathbb{G} - \mathbb{F}^2)} = 0, \tag{15}$$

$$\Gamma_{12}^2 = \frac{\mathbb{E}\,\mathbb{G}_u - \mathbb{F}\,\mathbb{E}_v}{2\,(\mathbb{E}\,\mathbb{G} - \mathbb{F}^2)} = 0, \tag{16}$$

$$\Gamma_{22}^1 = \frac{2\,\mathbb{G}\,\mathbb{F}_v - \mathbb{G}\,\mathbb{G}_u - \mathbb{F}\,\mathbb{G}_v}{2\,(\mathbb{E}\,\mathbb{G} - \mathbb{F}^2)} = \frac{48}{74} = \frac{24}{37}, \tag{17}$$

$$\Gamma_{22}^2 = \frac{\mathbb{E}\,\mathbb{G}_v - 2\,\mathbb{F}\,\mathbb{F}_v + \mathbb{F}\,\mathbb{G}_u}{2\,(\mathbb{E}\,\mathbb{G} - \mathbb{F}^2)} = 0. \tag{18}$$

QUESTION 3. (5 PTS) *Let S be the surface $z=x^2+y^2$. Let $\mathcal{C}$ be the intersection of S with the plane $z=1$. Let w be the tangent vector field $(-y, x)$ along $\mathcal{C}$. Is w parallel along $\mathcal{C}$? Justify your claim.*

Solution. We parametrize S by $(u,\, v,\, u^2+v^2)$ and $\mathcal{C}$ by $\gamma(t)=(\cos\, t,\, \sin\, t,\, 1)$. Thus $w(t)=(-\sin t, \cos t, 0)$. We further calculate

$$\sigma_u=(1,0,2\,u), \qquad \sigma_v=(0,1,2\,v) \tag{19}$$

and

$$N_S(u,v)=\frac{\sigma_u\times\sigma_v}{\|\sigma_u\times\sigma_v\|}=\frac{(-2\,u,-2\,v,1)}{\sqrt{1+4\,u^2+4\,v^2}}. \tag{20}$$

Thus along $\mathcal{C}$ we have

$$N_S(t)=\frac{(-2\cos t,-2\sin t,1)}{\sqrt{5}}. \tag{21}$$

Now we calculate

$$\begin{aligned}\nabla_\gamma w &= \dot{w}-(\dot{w}\cdot N_S)\,N_S\\ &= \begin{pmatrix}-\cos t\\-\sin t\\0\end{pmatrix}-\frac{1}{5}\left[\begin{pmatrix}-\cos t,\\-\sin t,\\0\end{pmatrix}\cdot\begin{pmatrix}-2\,\mathrm{cost}\\-2\sin t\\1\end{pmatrix}\right]\begin{pmatrix}-2\,\mathrm{cost}\\-2\sin t\\1\end{pmatrix}\\ &= \begin{pmatrix}-\cos t\\-\sin t\\0\end{pmatrix}-\frac{2}{5}\begin{pmatrix}-2\,\mathrm{cost}\\-2\sin t\\1\end{pmatrix}=-\frac{1}{5}\begin{pmatrix}\cos t\\\sin t\\2\end{pmatrix}\neq\begin{pmatrix}0\\0\\0\end{pmatrix}.\end{aligned} \tag{22}$$

Thus w is not parallel along $\mathcal{C}$.