Solutions to Homework 6

(Total 20 pts; Due Oct. 27 12pm)

Question 1. (10 pts) *Consider the surface patch* $\sigma(u, v) = (u, 2\,v, u\,v)$. *Calculate* H, K, $\kappa_1, \kappa_2, t_1, t_2$ *at* $p = (1, 2, 1)$.

Solution. $p = (1, 2, 1) = \sigma(1, 1)$.

1. We calculate the first fundamental form

$$\sigma_u = (1, 0, v) \overset{u=v=1}{=\!=\!=\!=} (1, 0, 1), \qquad \sigma_v = (0, 2, u) \overset{u=v=1}{=\!=\!=\!=} (0, 2, 1) \tag{1}$$

$$2\,\mathrm{d}u^2 + 2\,\mathrm{d}u\,\mathrm{d}v + 5\,\mathrm{d}v^2. \tag{2}$$

2. We calculate the second fundamental form

$$N = \frac{\sigma_u \times \sigma_v}{\|\sigma_u \times \sigma_v\|} = \frac{1}{3}(-2, -1, 2). \tag{3}$$

$$\sigma_{uu} = \sigma_{vv} \overset{u=v=1}{=\!=\!=\!=} 0, \qquad \sigma_{uv} = \sigma_{vu} \overset{u=v=1}{=\!=\!=\!=} (0, 0, 1). \tag{4}$$

Thus the second fundamental form is

$$\frac{4}{3}\, \mathrm{d}u\, \mathrm{d}v. \tag{5}$$

3. Principal curvatures solve

$$\det\left[\begin{pmatrix} 0 & \frac{2}{3} \\ \frac{2}{3} & 0 \end{pmatrix} - \lambda \begin{pmatrix} 2 & 1 \\ 1 & 5 \end{pmatrix}\right] = 0. \tag{6}$$

This becomes

$$10\,\lambda^2 - \left(\lambda - \frac{2}{3}\right)^2 = 0 \Longrightarrow \kappa_{1,2} = \frac{2}{3\left(1 \pm \sqrt{10}\right)}. \tag{7}$$

4. H and K.

We have

$$H = \frac{\kappa_1 + \kappa_2}{2} = -\frac{2}{27}; \qquad K = \kappa_1\,\kappa_2 = -\frac{4}{81}. \tag{8}$$

5. t_1 and t_2. Solve

$$\left[\begin{pmatrix} 0 & \frac{2}{3} \\ \frac{2}{3} & 0 \end{pmatrix} - \kappa_i \begin{pmatrix} 2 & 1 \\ 1 & 5 \end{pmatrix}\right] \begin{pmatrix} a_i \\ b_i \end{pmatrix} = 0. \tag{9}$$

We have

$$\begin{pmatrix} a_1 \\ b_1 \end{pmatrix} = c_1 \begin{pmatrix} \sqrt{5} \\ \sqrt{2} \end{pmatrix}, \qquad \begin{pmatrix} a_2 \\ b_2 \end{pmatrix} = c_2 \begin{pmatrix} -\sqrt{5} \\ \sqrt{2} \end{pmatrix}. \tag{10}$$

Requiring $\|t_1\| = \|t_2\| = 1$ we see that $c_1 = \left(20 + 2\sqrt{10}\right)^{-1/2}$ and $c_2 = \left(20 - 2\sqrt{10}\right)^{-1/2}$. Thus

$$t_1 = \left(20 + 2\sqrt{10}\right)^{-1/2} \left(\sqrt{5}\,\sigma_u + \sqrt{2}\,\sigma_v\right) = \left(20 + 2\sqrt{10}\right)^{-1/2} \begin{pmatrix} \sqrt{5} \\ 2\sqrt{2} \\ \sqrt{5} + 2\sqrt{2} \end{pmatrix}, \tag{11}$$

$$t_2 = \left(20 - 2\sqrt{10}\right)^{-1/2} \begin{pmatrix} -\sqrt{5} \\ 2\sqrt{2} \\ -\sqrt{5} + 2\sqrt{2} \end{pmatrix}. \tag{12}$$

QUESTION 2. (5 PTS) *Let γ be a curve in a surface S. Assume that at every $p \in \gamma$*

i. $\dot{\gamma}(p)$ is parallel to the principal vector t_1;

ii. the angle between the osculating plane and $T_p S$ is fixed;

iii. the normal curvature $\kappa_n \neq 0$.

Prove that γ is a plane curve. (Hint: Prove that $\dot{N}_S \parallel T$, then calculate $\frac{\mathrm{d}}{\mathrm{d}s}(N_S \cdot B)$.)

Proof. Let $\gamma(s) = \sigma(u(s), v(s))$ and let s be the arc length parameter. As $\dot{\gamma}$ is parallel to t_1, we have

$$\begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix} \begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \kappa_1 \begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} \tag{13}$$

which leads to (let $N_S(s)$ be the surface normal along γ)

$$-\dot{N}_S = (a_{11}\sigma_u + a_{12}\sigma_v)\,\dot{u} + (a_{21}\sigma_u + a_{22}\sigma_v)\dot{v} = \kappa_1\,(\sigma_u\,\dot{u} + \sigma_v\,\dot{v}) = \kappa_1\,\dot{\gamma}. \tag{14}$$

Thus

$$\dot{N}_S = -\kappa_1 T. \tag{15}$$

On the other hand, by assumption we have $N_S \cdot B = \text{constant}$. Therefore

$$0 = \frac{\mathrm{d}}{\mathrm{d}s}(N_S \cdot B) = -\kappa_1 T \cdot B - \tau N_S \cdot N = -\tau N_S \cdot N. \tag{16}$$

Now recall that

$$\kappa N = \kappa_n N_S + \kappa_g (N_S \times T) \Longrightarrow \kappa_n = \kappa N \cdot N_S. \tag{17}$$

Thus $\kappa_n \neq 0 \Longrightarrow \kappa \neq 0, N \cdot N_S \neq 0$. Consequently $\tau = 0$ and γ is therefore a plane curve. □

QUESTION 3. (5 PTS) *Let S_1, S_2 be two surfaces. Let the curve γ be their intersection. Let $p \in \gamma$. Let the normal curvatures at p of S_i along γ be $\kappa_n^{(i)}$, $i=1,2$. Let θ be the angle between the surface normals at p. Prove that*

$$\kappa^2 \sin^2\theta = \left(\kappa_n^{(1)}\right)^2 + \left(\kappa_n^{(2)}\right)^2 - 2\,\kappa_n^{(1)}\,\kappa_n^{(2)} \cos\theta. \tag{18}$$

(Hint: Prove that $\kappa_n^{(i)} = \kappa \cos\theta_i$)

Proof. Let N_i, $i=1,2$, be the surface normals. Let θ_i be the counter-clockwise angle from the curve normal N to N_i. We have

$$\kappa\, N = \kappa_n^{(i)}\, N_i + \kappa_g^{(i)}\, (N_i \times T) \Longrightarrow \kappa_n^{(i)} = \kappa\, N \cdot N_i = \kappa \cos\theta_i. \tag{19}$$

As $N_1, N_2, N \perp T$, the three normal vectors lies in the same plane.

Now let θ be the counter-clockwise angle from N_1 to N_2. Therefore by assumption $\theta_1 + \theta - \theta_2 = 2\,k\,\pi$ for some integer k.[1] Letting $\theta' := 2\,k\,\pi - \theta = \theta_1 - \theta_2$, we calculate (note that $\cos\theta' = \cos\theta$)

$$\begin{aligned}
\left(\kappa_n^{(1)}\right)^2 + \left(\kappa_n^{(2)}\right)^2 - 2\,\kappa_n^{(1)}\,\kappa_n^{(2)} \cos\theta &= \kappa^2 \left[\cos^2\theta_1 + \cos^2\theta_2 - 2\cos\theta_1 \cos\theta_2 \cos\theta'\right] \\
&= \kappa^2 \left[\cos\theta_1 \left(\cos(\theta_2 + \theta') - \cos\theta_2 \cos\theta'\right)\right] \\
&\quad + \kappa^2 \left[\cos\theta_2 \left(\cos(\theta_1 - \theta') - \cos\theta_1 \cos\theta'\right)\right] \\
&= \kappa^2 \left[-\cos\theta_1 \sin\theta_2 + \cos\theta_2 \sin\theta_1\right] \sin\theta' \\
&= \kappa^2 \sin^2\theta' = \kappa^2 \sin^2\theta.
\end{aligned} \tag{20}$$

In the last equality we have used $\sin\theta' = -\sin\theta$. □

1. From N to N_1 is θ_1, from N_1 to N_2 is θ, and from N_2 to N is $-\theta_2$. As we are back to N, the total angle must be an integer multiple of 2π.