Solutions to Homework 5

(Total 20 pts; Due Oct. 27 12pm)

Question 1. (5 pts) *Consider the surface patch* $\sigma(u,v) = (u^3\, v, u^2 + v^2, v)$. *Calculate its second fundamental form at* $p = (1,2,1)$.

Solution. We calculate

$$\sigma_u = (3\, u^2\, v, 2\, u, 0), \qquad \sigma_v = (u^3, 2\, v, 1), \tag{1}$$

$$\sigma_{uu} = (6\, u\, v, 2, 0), \quad \sigma_{uv} = (3\, u^2, 0, 0), \quad \sigma_{vv} = (0, 2, 0). \tag{2}$$

We note that at $p = (1,2,1) = \sigma(1,1)$, we have

$$\sigma_u = (3,2,0), \qquad \sigma_v = (1,2,1) \qquad \Longrightarrow \qquad N = \frac{\sigma_u \times \sigma_v}{\|\sigma_u \times \sigma_v\|} = \frac{(2,-3,4)}{\sqrt{29}}. \tag{3}$$

$$\sigma_{uu} = (6,2,0), \quad \sigma_{uv} = (3,0,0), \quad \sigma_{vv} = (0,2,0). \tag{4}$$

Thus

$$\mathbb{L} = \sigma_{uu} \cdot N = \frac{6}{\sqrt{29}}, \qquad \mathbb{M} = \sigma_{uv} \cdot N = \frac{6}{\sqrt{29}}, \qquad \mathbb{N} = \sigma_{vv} \cdot N = \frac{-6}{\sqrt{29}}, \tag{5}$$

and the second fundamental form is

$$\frac{6}{\sqrt{29}} (\mathrm{d}u^2 + 2\, \mathrm{d}u\, \mathrm{d}v - \mathrm{d}v^2). \tag{6}$$

QUESTION 2. (10 PTS) *Let S be a surface patch.*

a) (5 PTS) *Prove that the normal curvature at $p \in S$ in the direction $w \in T_p S$ is*

$$\kappa_n = \frac{\langle\langle w, w\rangle\rangle_{p,S}}{\langle w, w\rangle_{p,S}} \tag{7}$$

b) *Let the first and second fundamental forms of S be $(1 + v^2)\,\mathrm{d}u^2 + 2\,u\,v\,\mathrm{d}u\,\mathrm{d}v + (1 + u^2)\,\mathrm{d}v^2$ and $\frac{2\,\mathrm{d}u\,\mathrm{d}v}{\sqrt{1+u^2+v^2}}$ respectively. Calculate the normal curvature at point $\sigma(1,1)$ in the direction $\sigma_u + \sigma_v$.*

Solution.

a) We have

$$\begin{aligned}\kappa_n &= \langle\langle \frac{w}{\|w\|}, \frac{w}{\|w\|}\rangle\rangle_{p,S} \\ &= \frac{\langle\langle w, w\rangle\rangle_{p,S}}{\|w\|^2} \\ &= \frac{\langle\langle w, w\rangle\rangle_{p,S}}{\langle w, w\rangle_{p,S}}.\end{aligned}$$

b) At $(1,1)$ the first and second fundamental form read

$$2\,\mathrm{d}u^2 + 2\,\mathrm{d}u\,\mathrm{d}v + 2\,\mathrm{d}v^2, \qquad \frac{2}{\sqrt{3}}\,\mathrm{d}u\,\mathrm{d}v. \tag{8}$$

Now we have

$$\kappa_n = \frac{\langle\langle \sigma_u + \sigma_v, \sigma_u + \sigma_v\rangle\rangle}{\langle \sigma_u + \sigma_v, \sigma_u + \sigma_v\rangle} = \frac{\frac{2}{\sqrt{3}}\,1 \cdot 1}{2 \cdot 1^2 + 2 \cdot 1 \cdot 1 + 2 \cdot 1^2} = \frac{\sqrt{3}}{9}. \tag{9}$$

QUESTION 3. (5 PTS) *Let γ be a curve on a surface S. Let $p \in \gamma$. Assume that the osculating plane of γ at p coincides with $T_p S$. Prove that the normal curvature of S at p in the direction $\dot{\gamma}$ is zero.*

Proof. By assumption $N \perp N_S$. Since

$$\kappa N = \kappa_n N_S + \kappa_g (N_S \times T), \tag{10}$$

we see that

$$\kappa_n = (\kappa N) \cdot N_S = 0. \tag{11}$$

Thus ends the proof. $\square$