SOLUTIONS TO HOMEWORK 4

(TOTAL 20 PTS; DUE OCT. 20 12PM)

QUESTION 1. (5 PTS) *Consider the surface patch* $\sigma(u,v) = (u^3\, v, u^2 + v^2, v)$. *Calculate its first fundamental form at* $p = (1,2,1)$.

Solution. We have

$$\sigma_u = (3\,u^2\,v, 2\,u, 0), \qquad \sigma_v = (u^3, 2\,v, 1). \tag{1}$$

Thus

$$\begin{aligned} \mathbb{E} &= \sigma_u \cdot \sigma_u = 9\,u^4\,v^2 + 4\,u^2, \\ \mathbb{F} &= \sigma_u \cdot \sigma_v = 3\,u^5\,v + 4\,u\,v, \\ \mathbb{G} &= \sigma_v \cdot \sigma_v = u^6 + 4\,v^2 + 1. \end{aligned}$$

Now as $\sigma(1,1) = (1,2,1) = p$, we see that the first fundamental form at this point is

$$\mathbb{E}(1,1)\,\mathrm{d}u^2 + 2\,\mathbb{F}(1,1)\mathrm{d}u\,\mathrm{d}v + \mathbb{G}(1,1)\,\mathrm{d}v^2 = 13\,\mathrm{d}u^2 + 14\,\mathrm{d}u\,\mathrm{d}v + 6\,\mathrm{d}v^2. \tag{2}$$

Question 2. (10 pts) *Let S be a surface patch with first fundamental form $(1+v^2)\,\mathrm{d}u^2 + 2\,u\,v\,\mathrm{d}u\,\mathrm{d}v + (1+u^2)\,\mathrm{d}v^2$. Calculate the following.*

i. (4 pts) *The arc length of the curve $u=t, v=t$ for $0 \leqslant t \leqslant 1$.*

ii. (4 pts) *The angle between the curves $u=1$ and $v=1$.*

iii. (2 pts) *The area of $\sigma(U)$ where U is the region bounded by the positive u, v axes and the quarter circle $u^2+v^2=1$.*

Solution.

i. We have

$$\begin{aligned} L &= \int_0^1 \sqrt{\mathbb{E}(t,t)\,1^2 + 2\,\mathbb{F}(t,t)\,1\cdot 1 + \mathbb{G}(t,t)\,1^2}\,\mathrm{d}t \\ &= \int_0^1 \sqrt{(1+t^2) + 2\,t^2 + (1+t^2)}\,\mathrm{d}t \\ &= \int_0^1 \sqrt{2+4\,t^2}\,\mathrm{d}t \\ &= \left[t\sqrt{t^2+\frac{1}{2}} + \frac{1}{2}\ln\left(t + \sqrt{t^2+\frac{1}{2}} \right) \right]_0^1 \\ &= \sqrt{\frac{3}{2}} + \frac{1}{2}\ln\left(\sqrt{2}+\sqrt{3}\right). \end{aligned} \tag{3}$$

ii. The two curves intersect at $u=v=1$. We calculate

$$\mathbb{E}(1,1) = \mathbb{G}(1,1) = 2, \qquad \mathbb{F}(1,1) = 1. \tag{4}$$

Now we take the following parametrization of $u=1, v=1$: $(1,t), (t,1)$. That is we have $u_1(t)=1, v_1(t)=t$ and $u_2(t)=t$, $v_2(t)=1$. The intersection now is at $t_1=1, t_2=1$. We calculate

$$\dot{u}_1(t_1) = 0, \quad \dot{v}_1(t_1) = 1, \qquad \dot{u}_2(t_2) = 1, \quad \dot{v}_2(t_2) = 0. \tag{5}$$

We have

$$\cos\theta = \frac{1}{2} \Longrightarrow \theta = \frac{\pi}{3}. \tag{6}$$

iii. We obtain

$$\begin{aligned} A &= \int_U \sqrt{1+u^2+v^2}\,\mathrm{d}u\,\mathrm{d}v \\ &= \int_0^{\pi/2}\int_0^1 \sqrt{1+r^2}\,r\,\mathrm{d}r\,\mathrm{d}\theta = \frac{\pi}{6}\left(\sqrt{8}-1\right). \end{aligned} \tag{7}$$

Question 3. (5 pts) *Prove that the following is an equiareal mapping from the unit sphere to the plane:*

$$f(\cos u \cos v, \cos u \sin v, \sin u) = (u, v \cos u). \tag{8}$$

This is a mapping projection obtained by Sanson in 1650.

Solution. We have $\sigma_1(u, v) = (\cos u \cos v, \cos u \sin v, \sin u)$ and $\sigma_2(u, v) = (u, v \cos u)$. We calculate

$$\begin{aligned}
\sigma_{1,u} &= (-\sin u \cos v, -\sin u \sin v, \cos u); \\
\sigma_{1,v} &= (-\cos u \sin v, \cos u \cos v, 0); \\
\sigma_{2,u} &= (1, -v \sin u); \\
\sigma_{2,v} &= (0, \cos u).
\end{aligned}$$

Thus

$$\mathbb{E}_1 = 1, \quad \mathbb{F}_1 = 0, \quad \mathbb{G}_1 = \cos^2 u \Longrightarrow \mathbb{E}_1 \mathbb{G}_1 - \mathbb{F}_1^2 = \cos^2 u. \tag{9}$$

On the other hand

$$\mathbb{E}_2 = 1 + v^2 \sin^2 u, \quad \mathbb{F}_2 = -v \sin u \cos u, \quad \mathbb{G}_2 = \cos^2 u \tag{10}$$

which gives

$$\mathbb{E}_2 \mathbb{G}_2 - \mathbb{F}_2^2 = \cos^2 u. \tag{11}$$

Consequently the mapping is equiareal.