Solutions to Homework 3

(Total 20 pts; Due Oct. 13 12pm)

Question 1. (***10 pts***) *Let* $\gamma(t)=(t,t^2,t^3)$. *Calculate* $T(0), N(0), B(0), \kappa(0), \tau(0)$.

Solution. We have $\dot\gamma(0)=(1,0,0)$, $\ddot\gamma(0)=(0,2,0)$, $\dddot\gamma(0)=(0,0,6)$.

$$\begin{aligned}
T(0) &= \frac{\dot\gamma(0)}{\|\dot\gamma(0)\|}=(1,0,0);\\
B(0) &= \frac{(\dot\gamma(0)\times\ddot\gamma(0))}{\|\dot\gamma(0)\times\ddot\gamma(0)\|}=(0,0,1);\\
N(0) &= B(0)\times T(0)=(0,1,0);\\
\kappa(0) &= \frac{\|\dot\gamma(0)\times\ddot\gamma(0)\|}{\|\dot\gamma(0)\|^3}=2;\\
\tau(0) &= \frac{(\dot\gamma(0)\times\ddot\gamma(0))\cdot\dddot\gamma(0)}{\|\dot\gamma(0)\times\ddot\gamma(0)\|^2}=3.
\end{aligned}$$

QUESTION 2. (5 PTS) *Let γ be a curve such that its osculating plane (the plane passing $p \in \gamma$ and spanned by $T(p)$ and $N(p)$) passes a fixed point. Prove that γ is a planar curve.*

Solution. Denote the fixed point by γ_0. Then we have

$$B(s) \cdot (\gamma(s) - \gamma_0) = 0. \tag{1}$$

Differentiating this we obtain

$$\dot{B}(s) \cdot (\gamma(s) - \gamma_0) + B(s) \cdot T(s) = 0 \Longrightarrow -\tau N(s) \cdot (\gamma(s) - \gamma_0) = 0. \tag{2}$$

There are two cases.

i. $\tau = 0$. Then γ is planar.

ii. $\tau \neq 0$. Then

$$N(s) \cdot (\gamma(s) - \gamma_0) = 0. \tag{3}$$

Differentiating this we have

$$\dot{N}(s) \cdot (\gamma(s) - \gamma_0) + N(s) \cdot T(s) = 0 \tag{4}$$

which gives

$$(-\kappa T + \tau B) \cdot (\gamma(s) - \gamma_0) = 0. \tag{5}$$

Thanks to (1) we have

$$\kappa T \cdot (\gamma(s) - \gamma_0) = 0. \tag{6}$$

There are two cases.

a) $\kappa = 0$. Then γ is a straight line and is planar.

b) $\kappa \neq 0$. Then $T \cdot (\gamma(s) - \gamma_0) = 0$. Together with (1) and (3) we have $\gamma(s) - \gamma_0 = 0$ which means γ is a point, and is planar.

Question 3. (5 pts) *Prove that the two curves*

$$\gamma(t) = \left(t + \sqrt{3}\sin t, 2\cos t, \sqrt{3}\,t - \sin t\right) \text{ and } \tilde{\gamma}(u) = \left(2\cos\frac{u}{2}, 2\sin\frac{u}{2}, -u\right) \tag{7}$$

are congruent, that is there is a rigit motion M such that γ and $M\tilde{\gamma}$ coincide.

Solution. We have

$$\dot{\gamma} = \left(1 + \sqrt{3}\cos t, -2\sin t, \sqrt{3} - \cos t\right) \Longrightarrow \|\dot{\gamma}\| = 2\sqrt{2}, \tag{8}$$

$$\dot{\tilde{\gamma}}(u) = \left(-\sin\frac{u}{2}, \cos\frac{u}{2}, -1\right) \Longrightarrow \|\dot{\tilde{\gamma}}\| = \sqrt{2}. \tag{9}$$

Thus the arc length parametrizations of the two curves are

$$\gamma(s) = \left(\frac{s}{2\sqrt{2}} + \sqrt{3}\sin\frac{s}{2\sqrt{2}}, 2\cos\frac{s}{2\sqrt{2}}, \frac{\sqrt{3}}{2\sqrt{2}}\,s - \sin\frac{s}{2\sqrt{2}}\right) \tag{10}$$

and

$$\tilde{\gamma}(s) = \left(2\cos\frac{s}{2\sqrt{2}}, 2\sin\frac{s}{2\sqrt{2}}, -\frac{s}{\sqrt{2}}\right). \tag{11}$$

Now we calculate

$$\dot{\gamma}(s) = \left(\frac{1}{2\sqrt{2}} + \frac{\sqrt{3}}{2\sqrt{2}}\cos\frac{s}{2\sqrt{2}}, -\frac{1}{\sqrt{2}}\sin\frac{s}{2\sqrt{2}}, \frac{\sqrt{3}}{2\sqrt{2}} - \frac{1}{2\sqrt{2}}\cos\frac{s}{2\sqrt{2}}\right), \tag{12}$$

$$\ddot{\gamma}(s) = \left(-\frac{\sqrt{3}}{8}\sin\frac{s}{2\sqrt{2}}, -\frac{1}{4}\cos\frac{s}{2\sqrt{2}}, \frac{1}{8}\sin\frac{s}{2\sqrt{2}}\right) \Longrightarrow \kappa(s) = \frac{1}{4}, \tag{13}$$

$$\dddot{\gamma}(s) = \left(-\frac{\sqrt{3}}{16\sqrt{2}}\cos\frac{s}{2\sqrt{2}}, \frac{1}{8\sqrt{2}}\sin\frac{s}{2\sqrt{2}}, \frac{1}{16\sqrt{2}}\cos\frac{s}{2\sqrt{2}}\right) \Longrightarrow \tau(s) = -\frac{1}{4}. \tag{14}$$

and

$$\dot{\tilde{\gamma}} = \frac{1}{\sqrt{2}}\left(-\sin\frac{s}{2\sqrt{2}}, \cos\frac{s}{2\sqrt{2}}, -1\right), \tag{15}$$

$$\ddot{\tilde{\gamma}} = \frac{1}{4}\left(-\cos\frac{s}{2\sqrt{2}}, -\sin\frac{s}{2\sqrt{2}}, 0\right) \Longrightarrow \tilde{\kappa}(s) = \frac{1}{4}, \tag{16}$$

$$\dddot{\tilde{\gamma}} = \frac{1}{8\sqrt{2}}\left(\sin\frac{s}{2\sqrt{2}}, -\cos\frac{s}{2\sqrt{2}}, 0\right) \Longrightarrow \tilde{\tau}(s) = -\frac{1}{4}. \tag{17}$$

Thus we see that the two curves are congruent.