Solutions to Homework 2

(Total 20 pts; Due Sept. 29 12pm)

Question 1. (5 pts) *Let S be given by $\sigma(u,v)=(u,v,u\,v)$, $(u,v)\in\mathbb{R}^2$. Calculate T_pS, its tangent plane at $p=(1,2,2)$.*

Solution.

We have $p=\sigma(1,2)$. Thus

$$\begin{aligned}\sigma_u &= (1,0,v) \Longrightarrow \sigma_u(1,2)=(1,0,2);\\ \sigma_v &= (0,1,u) \Longrightarrow \sigma_v(1,2)=(0,1,1).\end{aligned}$$

Then

$$T_pS=\{a\,\sigma_u+b\,\sigma_v|\,a,b\in\mathbb{R}\}=\{(a,b,2\,a+b)|\,a,b\in\mathbb{R}\}. \qquad (1)$$

Question 2. (5 pts) *For the same S be given by $\sigma(u,v)=(u,v,u\,v)$, $(u,v)\in\mathbb{R}^2$. Calculate $\mathcal{G}(1,2,2)$ where $\mathcal{G}$ is the Gauss Map.*

Solution. We have

$$\sigma_u(1,2)=(1,0,2), \qquad \sigma_v(1,2)=(0,1,1). \tag{2}$$

Thus

$$\mathcal{G}(1,1,2)=N(1,2)=\frac{(1,0,2)\times(0,1,1)}{\|(1,0,2)\times(0,1,1)\|}=\frac{1}{\sqrt{6}}(-2,-1,1). \tag{3}$$

QUESTION 3. (5 PTS) *For the same S be given by $\sigma(u,v)=(u,v,u\,v)$, $(u,v)\in\mathbb{R}^2$. Let $\mathcal{G}$ be the Gauss map. Calculate $D_{(1,2,2)}\mathcal{G}(1,1,3)$.*

Solution. We have $(1,2,2)=\sigma(1,2)$ and $(1,1,3)=(1,0,2)+(0,1,1)=\sigma_u(1,2)+\sigma_v(1,2)$.

Next we calculate

$$\begin{aligned} N(u,v) &= \frac{\sigma_u\times\sigma_v}{\|\sigma_u\times\sigma_v\|} \\ &= \frac{(1,0,v)\times(0,1,u)}{\|(1,0,v)\times(0,1,u)\|} \\ &= \frac{(-v,-u,1)}{\sqrt{1+u^2+v^2}} \end{aligned}$$

and

$$N_u=\frac{(0,-1,0)}{\sqrt{1+u^2+v^2}}+\frac{(u\,v,u^2,-u)}{\sqrt{1+u^2+v^2}^3}, \qquad N_v=\frac{(-1,0,0)}{\sqrt{1+u^2+v^2}}+\frac{(v^2,u\,v,-v)}{\sqrt{1+u^2+v^2}^3} \tag{4}$$

which gives

$$N_u(1,2)=\frac{(2,-5,-1)}{6\sqrt{6}}, \qquad N_v(1,2)=\frac{(-2,2,-2)}{6\sqrt{6}}. \tag{5}$$

Consequently

$$D_{(1,2,2)}\mathcal{G}(1,1,3)=N_u(1,2)+N_v(1,2)=\frac{(0,-1,-1)}{2\sqrt{6}}. \tag{6}$$

Question 4. (5 pts) *Let $\sigma: \mathbb{R}^2 \mapsto S$ be a surface patch that is also an isometry. Prove that σ_u and σ_v are perpendicular.*

Proof. Let $\gamma(t) = (u(t), v(t))$, $t \in (\alpha, \beta)$ be an arbitrary curve in U. Then its image in S is $\Gamma(t) = \sigma(u(t), v(t))$. As σ is an isometry, we must have

$$\int_a^b \|\dot{\gamma}(t)\| \, \mathrm{d}t = \int_a^b \left\|\dot{\Gamma}(t)\right\| \mathrm{d}t \tag{7}$$

for all $\alpha < a < b < \beta$. Consequently

$$\|\dot{\gamma}(t)\| = \left\|\dot{\Gamma}(t)\right\|, \qquad \forall t \in (\alpha, \beta). \tag{8}$$

Next we calculate

$$\|\dot{\gamma}(t)\|^2 = \dot{u}(t)^2 + \dot{v}(t)^2, \tag{9}$$

$$\begin{aligned} \left\|\dot{\Gamma}(t)\right\|^2 &= \|\dot{u}(t)\, \sigma_u + \dot{v}(t)\, \sigma_v\|^2 \\ &= \dot{u}(t)^2 \, \|\sigma_u\|^2 + 2\, \dot{u}(t)\, \dot{v}(t)\, \sigma_u \cdot \sigma_v + \dot{v}(t)^2 \, \|\sigma_v\|^2. \end{aligned}$$

Consequently

$$\dot{u}(t)^2 + \dot{v}(t)^2 = \dot{u}(t)^2 \, \|\sigma_u\|^2 + 2\, \dot{u}(t)\, \dot{v}(t)\, \sigma_u \cdot \sigma_v + \dot{v}(t)^2 \, \|\sigma_v\|^2 \tag{10}$$

for all $(u(t), v(t))$.

Now without loss of generality, let $p = \sigma(0,0)$. We prove that $\sigma_u(0,0) \cdot \sigma_v(0,0) = 0$.

Taking $u(t) = t, v(t) = 0$ gives $\|\sigma_u(0,0)\| = 1$; Taking $u(t) = 0, v(t) = t$ gives $\|\sigma_v(0,0)\| = 1$. Finally, taking $u(t) = v(t) = t$ gives $\sigma_u(0,0) \cdot \sigma_v(0,0) = 0$. □