Solutions to Homework 1

(Total 20 pts; Due Sept. 22 12pm)

Question 1. (5 pts) *Let* $u = (3, 0, 4), v = (1, 0, 5), w = (0, 7, 0)$. *Calculate*

a) $\|u\|$; *b)* $u \cdot v$; *c)* $(u \times v) \cdot w$; *d)* $u \cdot (v \times w)$; *e) The angle between* u, w.

Solution.

We have

$$\|u\| = \sqrt{3^2 + 0^2 + 4^2} = 5; \tag{1}$$

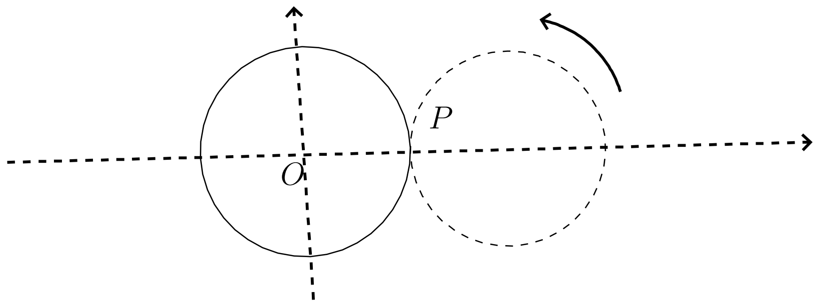
$$u \cdot v = 3 \cdot 1 + 0 \cdot 0 + 4 \cdot 5 = 23; \tag{2}$$

$$(u \times v) \cdot w = (0, 11, 0) \cdot (0, 7, 0) = 77; \tag{3}$$

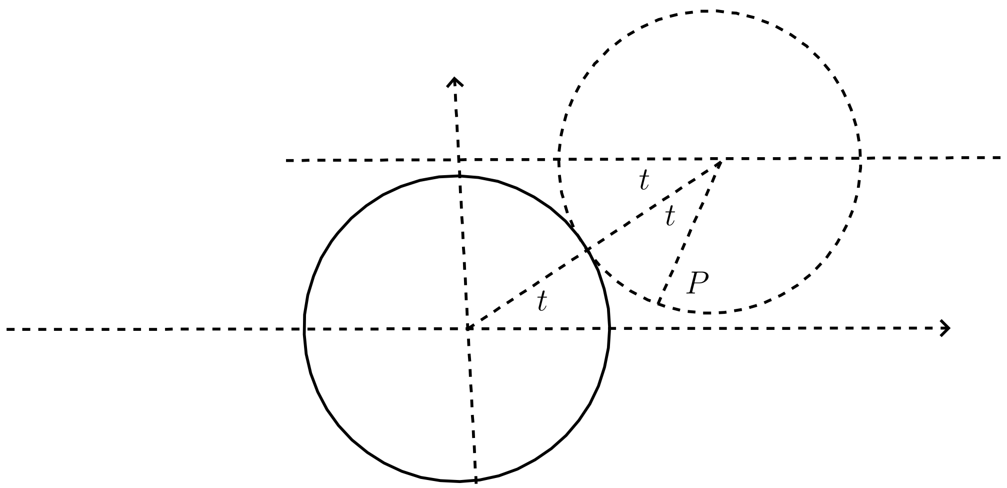
$$u \cdot (v \times w) = (u \times v) \cdot w = 77; \tag{4}$$

$$\cos\theta = \frac{u \cdot w}{\|u\| \, \|w\|} = 0 \Longrightarrow \theta = \pi/2. \tag{5}$$

Question 2. (5 pts) *Write down a parametrized representation of the trajectory of a fixed point P on a unit circle rolling outside another unit circle centered at the origin. Then calculate the arc length of the curve. (You may want to recall the formula for $\cos 2\theta$)*



Solution. Let t be the angle as shown in the plot below.



We see that the trajectory of P is given by

$$\gamma(t) = 2\,(\cos t, \sin t) - (\cos 2\,t, \sin 2\,t). \tag{6}$$

We have

$$\dot{\gamma}(t) = 2\,(-\sin t + \sin 2\,t, \cos t - \cos 2\,t) \tag{7}$$

which leads to

$$\|\dot{\gamma}(t)\| = 2\sqrt{2 - 2\cos t} = 2\sqrt{2 - 2\,(1 - 2\sin^2(t/2))} = 4\sin(t/2). \tag{8}$$

Note that here we have used the fact that $t \in (0, 2\,\pi)$ so $\sin(t/2) \geqslant 0$. Consequently the arc length is

$$L = \int_0^{2\pi} 4\sin(t/2)\,\mathrm{d}t = 16. \tag{9}$$

QUESTION 3. (5 PTS) *Let* $\gamma(t) = (5\cos t, 5\sin t, 12\,t)$. *Parametrize it by arc length.*

Solution. We have

$$\dot{\gamma}(t) = (-5\sin t, 5\cos t, 12) \Longrightarrow \|\dot{\gamma}(t)\| = 13. \tag{10}$$

Solving

$$\dot{S}(t) = \|\dot{\gamma}(t)\| = 13 \tag{11}$$

we obtain

$$s = S(t) = 13\,t \Longrightarrow t = T(s) = s/13. \tag{12}$$

The arc length parametrized curve now reads

$$\Gamma(s) = \left(5\cos\frac{s}{13}, 5\sin\frac{s}{13}, \frac{12}{13}\,s\right). \tag{13}$$

Question 4. (5 pts) *Calculate the surface area of*

$$S = \left\{ (x, y, z) | \, x^2 + y^2 + z^2 = 1, -\frac{1}{2} < z < \frac{1}{2} \right\}. \tag{14}$$

Solution. S can be parametrized as follows.

$$\sigma(u, v) = (\cos u \cos v, \cos u \sin v, \sin u), \qquad v \in (0, 2\pi), u \in \left(-\frac{\pi}{6}, \frac{\pi}{6} \right). \tag{15}$$

We calculate

$$\sigma_u = (-\sin u \cos v, -\sin u \sin v, \cos u), \qquad \sigma_v = (-\cos u \sin v, \cos u \cos v, 0), \tag{16}$$

and

$$\|\sigma_u \times \sigma_v\| = \cos u. \tag{17}$$

Consequently

$$A = \int_0^{2\pi} \int_{-\pi/6}^{\pi/6} \cos u \, \mathrm{d}u \, \mathrm{d}v = 2\pi. \tag{18}$$