Final Review II: Examples

We go through last year's final. This should not be treated as a "sample final".

Question 1. *Let* $\gamma(s) = \left(\frac{1}{\sqrt{2}}\cos s, \frac{1}{\sqrt{2}}\cos s, \sin s\right)$.

a) *Prove that s is the arc length parameter.*

b) *Calculate κ, τ, T, N, B.*

Solution.

a) We calculate

$$T = \dot{\gamma}(s) = \left(-\frac{\sin s}{\sqrt{2}}, -\frac{\sin s}{\sqrt{2}}, \cos s\right) \Longrightarrow \|\dot{\gamma}(s)\| = 1. \tag{1}$$

b) We have

$$\ddot{\gamma}(s) = \left(-\frac{\cos s}{\sqrt{2}}, -\frac{\cos s}{\sqrt{2}}, -\sin s\right), \tag{2}$$

which gives

$$\kappa = 1, \qquad N = \left(-\frac{\cos s}{\sqrt{2}}, -\frac{\cos s}{\sqrt{2}}, -\sin s\right). \tag{3}$$

We further have

$$\dddot{\gamma}(s) = \left(\frac{\sin s}{\sqrt{2}}, \frac{\sin s}{\sqrt{2}}, -\cos s\right). \tag{4}$$

Thus

$$\tau = \frac{(\dot{\gamma}(s) \times \ddot{\gamma}(s)) \cdot \dddot{\gamma}(s)}{\kappa(s)^2} = 0. \tag{5}$$

QUESTION 2. (5 PTS) *Let $\mathcal{C}$ be a regular curve with curvature $\kappa \neq 0$ at $p \in \mathcal{C}$. Let $\tilde{\mathcal{C}}$ be the orthogonal projection of $\mathcal{C}$ onto the osculating plane (the plane spanned by T and N) at p. Prove that $\mathcal{C}$ and $\tilde{\mathcal{C}}$ have the same curvature at p.*

Proof. Let $\mathcal{C}$ be parametrized as $\gamma(s)$ where s is the arc length parameter. Wlog $p = x(0)$. Denote $T(0), N(0), B(0)$, $\kappa(0)$, $\tau(0)$ by T_0, N_0, B_0, κ_0, τ_0.

Then $\tilde{\mathcal{C}}$ is given by

$$\tilde{\gamma}(s) = \gamma(s) - ((\gamma(s) - \gamma(0)) \cdot B_0)\, B_0. \tag{6}$$

Note that s in general is not the arc length parameter of $\tilde{\gamma}$.

Taking derivative we have

$$\begin{aligned} \dot{\tilde{\gamma}}(s) &= \dot{\gamma}(s) - (\dot{\gamma}(s) \cdot B_0)\, B_0 \\ &= T(s) - (T(s) \cdot B_0)\, B_0. \end{aligned}$$

Consequently $\dot{\tilde{\gamma}}(0) = T_0$.

Now by the Frenet-Serret equations

$$\ddot{\tilde{\gamma}}(s) = \kappa(s)\, [N(s) - (N(s) \cdot B_0)\, B_0] \Longrightarrow \ddot{\tilde{\gamma}}(0) = \kappa_0\, N_0 \tag{7}$$

Therefore

$$\begin{aligned} \tilde{\kappa}(0) &= \frac{\|\dot{\tilde{\gamma}}(0) \times \ddot{\tilde{\gamma}}(0)\|}{\|\dot{\tilde{\gamma}}(0)\|^3} \\ &= \kappa_0 = \kappa(0). \end{aligned} \tag{8}$$

Thus ends the proof. □

QUESTION 3. *Let S be a surface with first fundamental form $\mathrm{d}u^2 + (u^2+1)^2\,\mathrm{d}v^2$. Calculate*

i. The angle between the curves $u+v=0$ and $u-v=0$;

ii. The area of the curvilinear triangle bounded by $u=v, u=-v, v=1$.

Solution.

i. The intersection of the two curves is at $u=v=0$, where we have $\mathbb{E}=1, \mathbb{G}=1, \mathbb{F}=0$.

We parametrize:

$$u+v=0\text{:}\;\; u(t)=t, v(t)=-t; \tag{9}$$

$$u-v=0\text{:}\;\; U(t)=V(t)=t. \tag{10}$$

Thus

$$\begin{aligned}\cos\theta &= \frac{\left(\mathbb{E}\,\dot{u}\,\dot{U} + \mathbb{F}\left(\dot{u}\,\dot{V} + \dot{U}\,\dot{v}\right) + \mathbb{G}\,\dot{v}\,\dot{V}\right)}{\sqrt{\mathbb{E}\,\dot{u}^2 + 2\,\mathbb{F}\,\dot{u}\,\dot{v} + \mathbb{G}\,\dot{v}^2}\,\sqrt{\mathbb{E}\,\dot{U}^2 + 2\,\mathbb{F}\,\dot{U}\dot{V} + \mathbb{G}\,\dot{V}^2}}\\ &= \frac{1-1}{\sqrt{1+5}\,\sqrt{1+5}}\\ &= 0.\end{aligned} \tag{11}$$

Thus $\theta = \pi/2$.

ii. The region bounded by $u=v, u=-v, v=1$ is

$$\Omega = \{(u,v)|\, 0 \leqslant v \leqslant 1, -v \leqslant u \leqslant v\} = \{(u,v)|\, -1 \leqslant u \leqslant 1, 0 \leqslant v \leqslant |u|\}. \tag{12}$$

We have

$$\begin{aligned}\text{Area} &= \int_\Omega \sqrt{\mathbb{E}\,\mathbb{G} - \mathbb{F}^2}\,\mathrm{d}u\,\mathrm{d}v\\ &= \int_\Omega (u^2+1)\,\mathrm{d}u\,\mathrm{d}v\\ &= \int_0^1 \left[\int_{-v}^{v} (u^2+1)\,\mathrm{d}u\right]\mathrm{d}v\\ &= \int_0^1 \frac{2\,v^3}{3} + 2\,v\,\mathrm{d}v\\ &= \frac{7}{6}.\end{aligned}$$

QUESTION 4. (10 PTS) *Consider the surface patch* $\sigma(u,v) = (a\,(u+v), b\,(u-v), 2\,u\,v)$ *where* $a, b > 0$ *are constants. Calcualte at* $p = (0,0,0)$ *its principal curvatures, principal vectors, mean curvature, and Gaussian curvature.*

Solution. We see that $p = \sigma(0,0)$. Now calculate

$$\sigma_u = (a, b, 2\,v), \qquad \sigma_v = (a, -b, 2\,u) \tag{13}$$

which leads to

$$\mathbb{E} = a^2 + b^2 + 4\,v^2, \qquad \mathbb{F} = a^2 - b^2 + 4\,u\,v, \qquad \mathbb{G} = a^2 + b^2 + 4\,u^2. \tag{14}$$

Next we have

$$N = \frac{\sigma_u \times \sigma_v}{\|\sigma_u \times \sigma_v\|} = \frac{1}{\sqrt{b^2\,(u+v)^2 + a^2\,(u-v)^2 + a^2\,b^2}} \begin{pmatrix} b\,(u+v) \\ a\,(v-u) \\ -a\,b \end{pmatrix}, \tag{15}$$

and

$$\sigma_{uu} = (0,0,0), \quad \sigma_{uv} = (0,0,2), \quad \sigma_{vv} = (0,0,0). \tag{16}$$

At $(0,0)$ these become

$$\mathbb{E} = a^2 + b^2 = \mathbb{G}, \qquad \mathbb{F} = a^2 - b^2, \tag{17}$$

$$N = (0,0,-1), \qquad \sigma_{uu} = (0,0,0), \quad \sigma_{uv} = (0,0,2), \quad \sigma_{vv} = (0,0,0). \tag{18}$$

Consequently at $(0,0)$

$$\mathbb{L} = 0 = \mathbb{N}, \qquad \mathbb{M} = -2. \tag{19}$$

We solve

$$\det\left[\begin{pmatrix} 0 & -2 \\ -2 & 0 \end{pmatrix} - \kappa \begin{pmatrix} a^2+b^2 & a^2-b^2 \\ a^2-b^2 & a^2+b^2 \end{pmatrix}\right] = 0 \tag{20}$$

which gives

$$\kappa_1 = b^{-2}, \qquad \kappa_2 = -a^{-2}, \qquad H = \frac{a^2 - b^2}{2\,a^2\,b^2}, \qquad K = -a^{-2}\,b^{-2}. \tag{21}$$

For the principal vectors, we solve $\left[\begin{pmatrix} 0 & -2 \\ -2 & 0 \end{pmatrix} - \kappa_i \begin{pmatrix} a^2+b^2 & a^2-b^2 \\ a^2-b^2 & a^2+b^2 \end{pmatrix}\right]\begin{pmatrix} \alpha_i \\ \beta_i \end{pmatrix} = 0$ which gives $\begin{pmatrix} \alpha_1 \\ \beta_1 \end{pmatrix} = c\begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} \alpha_2 \\ \beta_2 \end{pmatrix} = c\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

$$t_1 = c\,[\sigma_u - \sigma_v] = c\begin{pmatrix} 0 \\ 2\,b \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \qquad t_2 = c\,(\sigma_u + \sigma_v) = c\begin{pmatrix} 2\,a \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}. \tag{22}$$

QUESTION 5. *Consider the surface patch* $\sigma(u,v)=(u\cos v, u\sin v, v)$.

a) *Calculate the Christoffel symbols* Γ_{ij}^k, $i,j,k=1,2$.

b) *Is* $u=1$ *a geodesic? Justify your claim.*

Solution.

a) We calculate

$$\sigma_u=(\cos v,\sin v,0),\qquad \sigma_v=(-u\sin v,u\cos v,1), \tag{23}$$

thus

$$\sigma_u\times\sigma_v=(\sin v,-\cos v,u). \tag{24}$$

Furthermore

$$\sigma_{uu}=(0,0,0),\qquad \sigma_{uv}=(-\sin v,\cos v,0),\qquad \sigma_{vv}=(-u\cos v,-u\sin v,0). \tag{25}$$

Thus clearly $\Gamma_{11}^1=\Gamma_{11}^2=0$. Furthermore

$$\Gamma_{12}^1=0,\qquad \Gamma_{12}^2=\frac{u}{u^2+1}, \tag{26}$$

$$\Gamma_{22}^1=-u,\qquad \Gamma_{22}^2=0. \tag{27}$$

b) We parametrize $u=1$ by arc length as $u(s)=1, v(s)=v(s)$. To find $v(s)$ we calculate

$$\mathbb{E}=1,\qquad \mathbb{F}=0,\qquad \mathbb{G}=1+u^2. \tag{28}$$

Thus $v(s)$ satisfies

$$1=\mathbb{G}\,\dot{v}^2=2\,\dot{v}^2 \tag{29}$$

along the curve. Consequently we can take $v(s)=s/\sqrt{2}$.

Now along the curve $u=1$ we have

$$\Gamma_{12}^2=\frac{1}{2},\qquad \Gamma_{22}^1=-1, \tag{30}$$

and other Γ_{ij}^k's are all zero. The geodesic equations now read

$$0=\ddot{u}-\dot{v}^2=-\frac{1}{2}, \tag{31}$$

$$0=\ddot{v}+\dot{u}\,\dot{v}=0. \tag{32}$$

We see that (31) is not satisfied. So $u=1$ is not a geodesic.

QUESTION 6. *Let S be a surface with non-positive Gaussian curvature. Prove that there cannot be a smooth closed geodesic that bounds a simply connected region.*

Proof. Assume there is such a curve. Denote it by $\mathcal{C}$ and the region enclosed by Ω. Gauss-Bonnet now gives

$$2\pi = \int_\Omega K + \int_{\mathcal{C}} \kappa_g \leqslant \int_{\mathcal{C}} \kappa_g = 0. \tag{33}$$

Contradiction. □

QUESTION 7. *Let γ be a space curve. Let S be the ruled surface generated by γ and its binormals. Prove that γ is a geodesic on S.*

Proof. (SHORT) The tangent plane of S is span$\{T, B\}$ thus $\dot{T} \parallel N \perp T_p S$. Consequently $\nabla_\gamma T = 0$. $\square$

Proof. (LONG) We parametrize γ by its arc length u. Then S is given by the surface patch

$$\sigma(u, v) = \gamma(u) + v\,B(u). \tag{34}$$

We have

$$\sigma_u = T - \tau v\,N, \qquad \sigma_v = B \tag{35}$$

and

$$\sigma_{uu} = \kappa\,N - \tau v\,(-\kappa\,T + \tau B) = \kappa\,\tau v\,T + \kappa\,N - \tau^2\,v\,B, \quad \sigma_{uv} = -\tau N, \quad \sigma_{vv} = 0. \tag{36}$$

Next we have

$$\sigma_u \times \sigma_v = -N - \tau v\,T \Longrightarrow N_S = \frac{-N - \tau v\,T}{\sqrt{1 + \tau^2 v^2}}. \tag{37}$$

Thus

$$\mathbb{L} = -\kappa\sqrt{1 + \tau^2 v^2}, \qquad \mathbb{M} = \frac{\tau}{\sqrt{1 + \tau^2 v^2}}, \qquad \mathbb{N} = 0. \tag{38}$$

We then solve

$$\kappa\,\tau v\,T + \kappa\,N - \tau^2\,v\,B = \sigma_{uu} = \Gamma_{11}^1 (T - \tau v\,N) + \Gamma_{11}^2\,B + \kappa\,(N + \tau v\,T) \tag{39}$$

which gives

$$\kappa\,\tau v = \Gamma_{11}^1 + \kappa\,\tau v, \tag{40}$$

$$\kappa = -\Gamma_{11}^1\,\tau v + \kappa, \tag{41}$$

$$-\tau^2\,v = \Gamma_{11}^2, \tag{42}$$

so

$$\Gamma_{11}^1 = 0, \qquad \Gamma_{11}^2 = -\tau^2\,v. \tag{43}$$

Similarly

$$-\tau N = \Gamma_{12}^1 (T - \tau v\,N) + \Gamma_{12}^2\,B - \tau\,\frac{N + \tau v\,T}{1 + \tau^2 v^2} \Longrightarrow \Gamma_{12}^1 = \frac{\tau^2\,v}{1 + \tau^2 v^2}, \quad \Gamma_{12}^2 = 0, \tag{44}$$

and $\Gamma_{22}^1 = \Gamma_{22}^2 = 0$.

The unit tangent vector T is given by $\dot{u}(t)\,\sigma_u + \dot{v}(t)\,\sigma_v$ with $u(t) = t, v(t) = 0$. Note that by our setup t is the arc length here. We check

$$\ddot{u} + (\dot{u}, \dot{v})\left(\,\Gamma_{ij}^1\,\right)\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = 0 + \Gamma_{11}^1 = 0, \quad \ddot{v} + (\dot{u}, \dot{v})\left(\,\Gamma_{ij}^2\,\right)\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = 0 + \Gamma_{11}^2 = 0. \tag{45}$$

as $\Gamma_{11}^2 = -\tau^2\,v = 0$ along the curve $u(t) = t, v(t) = 0$. $\square$