Midterm 2

(Nov. 3, 2016, 11am–12pm. Total 15+3 pts)

NAME: ______________________________ ID#: ______________________________

- There are four problems (total 15 pts + 3 bonus pts).
- Please write clearly and show enough work.
- The following formulas may be useful:
  - Geodesic equations.

$$u'' + \begin{pmatrix} u' & v' \end{pmatrix} \begin{pmatrix} \Gamma_{11}^1 & \Gamma_{12}^1 \\ \Gamma_{12}^1 & \Gamma_{22}^1 \end{pmatrix} \begin{pmatrix} u' \\ v' \end{pmatrix} = 0, \tag{1}$$

$$v'' + \begin{pmatrix} u' & v' \end{pmatrix} \begin{pmatrix} \Gamma_{11}^2 & \Gamma_{12}^2 \\ \Gamma_{12}^2 & \Gamma_{22}^2 \end{pmatrix} \begin{pmatrix} u' \\ v' \end{pmatrix} = 0. \tag{2}$$

QUESTION 1. (5 PTS) *Let S be a surface with surface patch $\sigma(u,v) := (u, v, e^{uv})$.*

a) (3 PTS) *Calculate its first fundamental form.*

b) (2 PTS) *Calculate* $\cos\theta$ *where* θ *is the angle between the two curves* $u = v$ *and* $u = 1$.

Solution.

a) We calculate

$$\sigma_u = (1, 0, e^{uv} v), \qquad \sigma_v = (0, 1, e^{uv} u). \tag{3}$$

Therefore

$$\mathbb{E} = 1 + e^{2uv} v^2, \qquad \mathbb{F} = e^{2uv} u\, v, \qquad \mathbb{G} = 1 + e^{2uv} u^2 \tag{4}$$

and the first fundamental form is

$$(1 + e^{2uv} v^2)\, \mathrm{d}u^2 + 2\, e^{2uv} u\, v\, \mathrm{d}u\, \mathrm{d}v + (1 + e^{2uv} u^2)\, \mathrm{d}v^2. \tag{5}$$

b) We parametrize

- $u = v$: $x(t) = \sigma(u(t), v(t))$, $u(t) = t, v(t) = t$;
- $u = 1$: $\tilde{x}(\tilde{t}) = \sigma(\tilde{u}(\tilde{t}), \tilde{v}(\tilde{t}))$, $\tilde{u}(\tilde{t}) = 1, \tilde{v}(\tilde{t}) = \tilde{t}$.

Thus the intersection is at $x(1) = \tilde{x}(1) = \sigma(1, 1)$. We calculate

$$u'(1) = 1, \quad v'(1) = 1, \quad \tilde{u}'(1) = 0, \quad \tilde{v}'(1) = 1 \tag{6}$$

and

$$\mathbb{E}(p) = 1 + e^2, \qquad \mathbb{F}(p) = e^2, \qquad \mathbb{G}(p) = 1 + e^2. \tag{7}$$

Therefore

$$\begin{aligned}
\cos\theta &= \frac{\mathbb{E}\, u'\, \tilde{u}' + \mathbb{F}\, (u'\, \tilde{v}' + \tilde{u}'\, v') + \mathbb{G}\, v'\, \tilde{v}'}{\sqrt{\mathbb{E}\, u'^2 + 2\, \mathbb{F}\, u'\, v' + \mathbb{G}\, v'^2}\, \sqrt{\mathbb{E}\, \tilde{u}'^2 + 2\, \mathbb{F}\, \tilde{u}'\, \tilde{v}' + \mathbb{G}\, \tilde{v}'^2}} \\
&= \frac{(1 + e^2) \cdot 1 \cdot 0 + e^2\, (1 \cdot 1 + 1 \cdot 0) + (1 + e^2) \cdot 1 \cdot 1}{\sqrt{(1 + e^2)\, 1^2 + 2\, e^2\, 1 \cdot 1 + (1 + e^2)\, 1^2}\, \sqrt{(1 + e^2)\, 0^2 + 2\, e^2\, 1 \cdot 0 + (1 + e^2)\, 1^2}} \\
&= \frac{1 + 2\, e^2}{\sqrt{2 + 4\, e^2}\, \sqrt{1 + e^2}} = \sqrt{\frac{1 + 2\, e^2}{2 + 2\, e^2}}.
\end{aligned} \tag{8}$$

QUESTION 2. (5 PTS) *Consider the same surface patch as in Question 1,* $\sigma(u,v) := (u, v, e^{uv})$. *Let* $p = (0, 0, 1)$. *Calculate the principal curvatures* $\kappa_{1,2}$, *the mean curvature* H, *and the Gaussian curvature* K *at* p.

Solution.

From Question 1 we have

$$\mathbb{E} = 1 + e^{2uv} v^2, \qquad \mathbb{F} = e^{2uv} u\, v, \qquad \mathbb{G} = 1 + e^{2uv} u^2 \tag{9}$$

At $(0,0,1) = \sigma(0,0)$ we have $\mathbb{E} = 1, \mathbb{F} = 0, \mathbb{G} = 1$.

We further calculate, at p,

$$N = \frac{\sigma_u \times \sigma_v}{\|\sigma_u \times \sigma_v\|} = (0,0,1). \tag{10}$$

Next calculate

$$\sigma_{uu} = (0, 0, e^{uv} v^2), \qquad \sigma_{uv} = (0, 0, e^{uv} (1 + u\, v)), \qquad \sigma_{vv} = (0, 0, e^{uv} u^2). \tag{11}$$

At p we have

$$\sigma_{uu} = (0,0,0), \qquad \sigma_{uv} = (0,0,1), \qquad \sigma_{vv} = (0,0,0). \tag{12}$$

Therefore at p,

$$\mathbb{L} = 0, \qquad \mathbb{M} = 1, \qquad \mathbb{N} = 0. \tag{13}$$

We solve for $\kappa_{1,2}$:

$$0 = \det\left[\begin{pmatrix} \mathbb{L} & \mathbb{M} \\ \mathbb{M} & \mathbb{N} \end{pmatrix} - \kappa \begin{pmatrix} \mathbb{E} & \mathbb{F} \\ \mathbb{F} & \mathbb{G} \end{pmatrix}\right] = \det\begin{pmatrix} -\kappa & 1 \\ 1 & -\kappa \end{pmatrix}, \tag{14}$$

and obtain

$$\kappa_1 = 1, \qquad \kappa_2 = -1. \tag{15}$$

Therefore $H = 0, K = -1$.

QUESTION 3. (5 PTS) *Consider the same surface patch as in Questions 1 and 2, $\sigma(u,v):=(u, v, e^{uv})$.*

a) (3 PTS) *Calculate the Christoffel symbols Γ_{ij}^k.*

b) (2 PTS) *Is $u=0$ a geodesic? Justify your claim.*

Solution.

a) We calculate

$$\sigma_u \times \sigma_v = (-e^{uv} v, -e^{uv} u, 1). \tag{16}$$

Therefore

$$\begin{pmatrix} 0 \\ 0 \\ e^{uv} v^2 \end{pmatrix} = \sigma_{uu} = \Gamma_{11}^1 \begin{pmatrix} 1 \\ 0 \\ e^{uv} v \end{pmatrix} + \Gamma_{11}^2 \begin{pmatrix} 0 \\ 1 \\ e^{uv} u \end{pmatrix} + l \begin{pmatrix} -e^{uv} v \\ -e^{uv} u \\ 1 \end{pmatrix}. \tag{17}$$

We see that $\Gamma_{11}^1 = l\, e^{uv} v, \Gamma_{11}^2 = l\, e^{uv} u$. Substituting into the third equation we have

$$e^{uv} v^2 = l\, e^{2uv} v^2 + l\, e^{2uv} u^2 + l \Longrightarrow l = \frac{e^{uv} v^2}{e^{2uv} (u^2 + v^2) + 1}. \tag{18}$$

Therefore

$$\Gamma_{11}^1 = \frac{e^{2uv} v^3}{e^{2uv} (u^2 + v^2) + 1}, \qquad \Gamma_{11}^2 = \frac{e^{2uv} v^2 u}{e^{2uv} (u^2 + v^2) + 1}. \tag{19}$$

Next we have

$$\begin{pmatrix} 0 \\ 0 \\ e^{uv} (1 + u v) \end{pmatrix} = \sigma_{uv} = \Gamma_{12}^1 \begin{pmatrix} 1 \\ 0 \\ e^{uv} v \end{pmatrix} + \Gamma_{12}^2 \begin{pmatrix} 0 \\ 1 \\ e^{uv} u \end{pmatrix} + m \begin{pmatrix} -e^{uv} v \\ -e^{uv} u \\ 1 \end{pmatrix} \tag{20}$$

which gives

$$\Gamma_{12}^1 = \frac{e^{2uv} (1 + u v) v}{e^{2uv} (u^2 + v^2) + 1}, \qquad \Gamma_{12}^2 = \frac{e^{2uv} (1 + u v) u}{e^{2uv} (u^2 + v^2) + 1}. \tag{21}$$

Finally we calculate

$$\begin{pmatrix} 0 \\ 0 \\ e^{uv} u^2 \end{pmatrix} = \sigma_{vv} = \Gamma_{22}^1 \begin{pmatrix} 1 \\ 0 \\ e^{uv} v \end{pmatrix} + \Gamma_{22}^2 \begin{pmatrix} 0 \\ 1 \\ e^{uv} u \end{pmatrix} + n \begin{pmatrix} -e^{uv} v \\ -e^{uv} u \\ 1 \end{pmatrix} \tag{22}$$

which gives

$$\Gamma_{22}^1 = \frac{e^{2uv} v u^2}{e^{2uv} (u^2 + v^2) + 1}, \qquad \Gamma_{22}^2 = \frac{e^{2uv} u^3}{e^{2uv} (u^2 + v^2) + 1}. \tag{23}$$

b) We parametrize $u = 0$ as $u(t) = 0, v(t) = t$. Note that $x(t) := \sigma(u(t), v(t)) = (0, t, 1)$ is arc length parametrized.

Next along $u = 0$, we have

$$\begin{aligned} \Gamma_{11}^1 &= \frac{v^3}{1 + v^2}, \quad \Gamma_{11}^2 = 0, \\ \Gamma_{12}^1 &= \frac{v}{1 + v^2}, \quad \Gamma_{12}^2 = 0, \\ \Gamma_{22}^1 &= 0, \qquad\quad \Gamma_{22}^2 = 0. \end{aligned} \tag{24}$$

Therefore the geodesic equations are satisfied along $u = 0$:

$$0 + \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{v^3}{1+v^2} & 0^1 \\ 0^1 & 0^1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0, \tag{25}$$

$$0 + \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0. \tag{26}$$

So $u = 0$ is a geodesic.

Question 4. (Bonus. 3 pts) *Let S be a surface such that for every $p, q \in S$, the parallel transport map Π_γ^{pq} is independent of the path γ connecting p, q. Prove or disprove: S is part of a plane.*

Proof. For cylinder we have $\Gamma_{ij}^k = 0$. Thus the parallel transport equation reads

$$\alpha' = \beta' = 0. \tag{27}$$

Thus the conditions are satisfied and the claim disproved.

In fact S can be any surface with the Riemann curvature tensor $R_{ijk}^l = 0$ for all l, i, j, k. $\square$