Review for Midterm 2: Theory of Surfaces

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Warning.

This review only covers the basics. You should also go over the notes for lectures 8–15 as well as the sections in the textbook as listed on the course website.

- Let S be a surface parametrized by $\sigma(u,v)$.
- $p=\sigma(u_0,v_0)\in S$.

1. Measurement on surfaces

1.1. First fundamental form

- First fundamental form: $w,\tilde{w}\in T_pS$, then

$$\langle w,\tilde{w}\rangle_{p,S}:=w\cdot\tilde{w} \tag{1}$$

where the inner product on the right hand side is the inner product in $\mathbb{R}^3$.

- Restricted to T_pS (forget about the ambient $\mathbb{R}^3$), $w=\alpha\,\sigma_u+\beta\,\sigma_v$, $\tilde{w}=\tilde{\alpha}\,\sigma_u+\tilde{\beta}\,\sigma_v$, then

$$\langle w,\tilde{w}\rangle_{p,S}=\mathbb{E}(u_0,v_0)\,\alpha\,\tilde{\alpha}+\mathbb{F}(u_0,v_0)\,(\alpha\,\tilde{\beta}+\tilde{\alpha}\,\beta)+\mathbb{G}(u_0,v_0)\,\beta\,\tilde{\beta}. \tag{2}$$

- Alternative notation of first fundamental form.

$$\mathbb{E}(u,v)\,\mathrm{d}u^2+2\,\mathbb{F}(u,v)\,\mathrm{d}u\,\mathrm{d}v+\mathbb{G}(u,v)\,\mathrm{d}v^2. \tag{3}$$

- Calculation.

$$\mathbb{E}(u,v) := \|\sigma_u(u,v)\|^2, \tag{4}$$
$$\mathbb{F}(u,v) := \sigma_u(u,v)\cdot\sigma_v(u,v), \tag{5}$$
$$\mathbb{G}(u,v) := \|\sigma_v(u,v)\|^2. \tag{6}$$

1.2. Measurement using the first fundamental form

- Arc length for the curve $x(t):=\sigma(u(t),v(t))$ from $t=a$ to $t=b$.

$$L=\int_a^b\sqrt{\mathbb{E}(x(t))\,u'(t)^2+2\,\mathbb{F}(x(t))\,u'(t)\,v'(t)+\mathbb{G}(x(t))\,v'(t)^2}\,\mathrm{d}t. \tag{7}$$

- Angle between $x_1(t):=\sigma(u_1(t),\,v_1(t))$ and $x_2(t):=\sigma(u_2(t),\,v_2(t))$. Assume the two curves intersect at $p=\sigma(u_0,v_0)=x_1(t_1)=x_2(t_2)$.

$$\cos\theta=\frac{\mathbb{E}\,u_1'(t_1)\,u_2'(t_2)+\mathbb{F}\,(u_1'(t_1)\,v_2'(t_2)+u_2'(t_1)\,v_1'(t_2))+\mathbb{G}v_1'(t_1)\,v_2'(t_2)}{\sqrt{\mathbb{E}\,u_1'(t_1)^2+2\,\mathbb{F}\,u_1'(t_1)\,v_1'(t_1)+\mathbb{G}\,v_1'(t_1)^2}\,\sqrt{\mathbb{E}\,u_2'(t_2)^2+2\,\mathbb{F}\,u_2'(t_2)\,v_2'(t_2)+\mathbb{G}\,v_2'(t_2)^2}}. \tag{8}$$

Here $\mathbb{E}=\mathbb{E}(u_0,v_0)$, $\mathbb{F}=\mathbb{F}(u_0,v_0)$, $\mathbb{G}=\mathbb{G}(u_0,v_0)$.

- Area of $\sigma(U)$.

$$\int_U\sqrt{\mathbb{E}(\sigma(u,v))\,\mathbb{G}(\sigma(u,v))-\mathbb{F}(\sigma(u,v))^2}\,\mathrm{d}u\,\mathrm{d}v. \tag{9}$$

2. Curvatures

2.1. Second fundamental form

- Second fundamental form: $w, \tilde{w} \in T_pS$, $w = \alpha\,\sigma_u + \beta\,\sigma_v$, $\tilde{w} = \tilde{\alpha}\,\sigma_u + \tilde{\beta}\,\sigma_v$, then we define the second fundamental form as

$$\langle\langle w, \tilde{w}\rangle\rangle_{p,S} = \mathbb{L}(u_0, v_0)\,\alpha\,\tilde{\alpha} + 2\,\mathbb{M}(u_0, v_0)\,(\alpha\,\tilde{\beta} + \tilde{\alpha}\,\beta) + \mathbb{N}(u_0, v_0)\,\beta\,\tilde{\beta}. \tag{10}$$

  Alternative notation:

$$\mathbb{L}(u, v)\,\mathrm{d}u^2 + 2\,\mathbb{M}(u, v)\,\mathrm{d}u\,\mathrm{d}v + \mathbb{N}(u, v)\,\mathrm{d}v^2. \tag{11}$$

- Geometric meaning.
  - Gauss map.

$$p \in S \mapsto \mathcal{G}(p) \text{ the unit normal at } p. \tag{12}$$

    Alternatively,

$$\mathcal{G}(\sigma(u, v)) = N(u, v). \tag{13}$$

  - Weingarten map.

$$\mathcal{W}_{p,S} := -D\mathcal{G}. \tag{14}$$

    More specifically,

$$\mathcal{W}(\alpha\,\sigma_u + \beta\,\sigma_v) = -\alpha\,N_u - \beta\,N_v. \tag{15}$$

- Calculation. Let N be the unit normal of S at p.

$$\begin{aligned} \mathbb{L} &:= \sigma_{uu} \cdot N = -\sigma_u \cdot N_u, & (16)\\ \mathbb{M} &:= \sigma_{uv} \cdot N = -\sigma_u \cdot N_v = -\sigma_v \cdot N_u, & (17)\\ \mathbb{N} &:= \sigma_{vv} \cdot N = -\sigma_v \cdot N_v. & (18) \end{aligned}$$

- Relation to the first fundamental form.
  - Abstract:

$$\langle\langle w, \tilde{w}\rangle\rangle_{p,S} = \langle \mathcal{W}_{p,S}(w), \tilde{w}\rangle_{p,S} = \langle w, \mathcal{W}_{p,S}(\tilde{w})\rangle_{p,S}. \tag{19}$$

  - Concrete:

$$-N_u = a_{11}\,\sigma_u + a_{12}\,\sigma_v, \qquad -N_v = a_{21}\,\sigma_u + a_{22}\,\sigma_v, \tag{20}$$

$$\left(\begin{array}{cc} a_{11} & a_{21}\\ a_{12} & a_{22} \end{array}\right) = \left(\begin{array}{cc} \mathbb{E} & \mathbb{F}\\ \mathbb{F} & \mathbb{G} \end{array}\right)^{-1} \left(\begin{array}{cc} \mathbb{L} & \mathbb{M}\\ \mathbb{M} & \mathbb{N} \end{array}\right). \tag{21}$$

  - What is $\left(\begin{smallmatrix} a_{11} & a_{21}\\ a_{12} & a_{22} \end{smallmatrix}\right)$. We have

$$\begin{aligned} \mathcal{W}_{p,S}(\alpha\,\sigma_u + \beta\,\sigma_v) &= -\alpha\,N_u - \beta\,N_v\\ &= \alpha\,(a_{11}\,\sigma_u + a_{12}\,\sigma_v) + \beta\,(a_{21}\,\sigma_u + a_{22}\,\sigma_v)\\ &= (a_{11}\,\alpha + a_{21}\,\beta)\,\sigma_u + (a_{12}\,\alpha + a_{22}\,\beta)\,\sigma_v. \qquad (22) \end{aligned}$$

Thus we see that

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} \mapsto \begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix}. \tag{23}$$

So $\left(\begin{smallmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{smallmatrix}\right)$ is the matrix representation of the Weingarten map with respect to the basis $\{\sigma_u, \sigma_v\}$.

2.2. Curvatures

- Normal curvature. $w \in T_p S$.
  - Curvature of the intersection between S and the plane spanned by $\{w, N\}$.
  - Calculation of the normal curvature of S at p along w is

$$\kappa_n(w) = \frac{\langle\langle w, w\rangle\rangle_{p,S}}{\langle w, w\rangle_{p,S}}. \tag{24}$$

- Principal curvatures.
  - κ_1: maximal normal curvature; κ_2: minimal normal curvature.
  - Principal directions: $t_1 \perp t_2$.
  - $(\kappa_1, t_1), (\kappa_2, t_2)$: (Eigenvalue, eigenvector) of $\mathcal{W}_{p,S}$.
  - Calculation.

$$\det\begin{pmatrix} \mathbb{L} - \kappa_i \mathbb{E} & \mathbb{M} - \kappa_i \mathbb{F} \\ \mathbb{M} - \kappa_i \mathbb{F} & \mathbb{N} - \kappa_i \mathbb{G} \end{pmatrix} = 0, \tag{25}$$

$$\left[\begin{pmatrix} \mathbb{L} & \mathbb{M} \\ \mathbb{M} & \mathbb{N} \end{pmatrix} - \kappa_i \begin{pmatrix} \mathbb{E} & \mathbb{F} \\ \mathbb{F} & \mathbb{G} \end{pmatrix}\right] \begin{pmatrix} a_i \\ b_i \end{pmatrix} = 0, \tag{26}$$

$$t_i = a_i \sigma_u + b_i \sigma_v. \tag{27}$$

- Mean curvature.
  - Average of κ_n over all directions. $H = \frac{1}{2\pi} \int_0^{2\pi} \kappa_n(\theta)\, \mathrm{d}\theta$.
  - Calculation.

$$H = \frac{\mathbb{E}\mathbb{N} + \mathbb{L}\mathbb{G} - 2\,\mathbb{M}\mathbb{F}}{2\,(\mathbb{E}\mathbb{G} - \mathbb{F}^2)} = \mathrm{Tr}\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}. \tag{28}$$

- Gaussian curvature.

$$K = \lim_{r \longrightarrow 0} \frac{\text{Area of } N(B_r)}{\text{Area of } \sigma(B_r)} = \frac{\mathbb{L}\mathbb{N} - \mathbb{M}^2}{\mathbb{E}\mathbb{G} - \mathbb{F}^2} = \det\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}. \tag{29}$$

- Relations between curvatures.

$$H = \frac{\kappa_1 + \kappa_2}{2}, \qquad K = \kappa_1 \kappa_2, \qquad \kappa_{1,2} = \frac{H \pm \sqrt{H^2 - 4K}}{2}. \tag{30}$$

$$\kappa_n((\cos\theta)\, t_1 + (\sin\theta)\, t_2) = \kappa_1 \cos^2\theta + \kappa_2 \sin^2\theta. \tag{31}$$

2.3. Geodesics

- A parametrized curve $x(t)$ on the surface S is called a geodesic if $\nabla_\gamma x'(t) = 0$.

- Geodesic equations: $x(t) = \sigma(u(t), v(t))$.

$$\begin{aligned}(\mathbb{E}\,u' + \mathbb{F}\,v')' = (x'(t)\cdot\sigma_u)' &= \frac{1}{2}\begin{pmatrix} u' & v' \end{pmatrix}\begin{pmatrix} \mathbb{E} & \mathbb{F} \\ \mathbb{F} & \mathbb{G} \end{pmatrix}_u \begin{pmatrix} u' \\ v' \end{pmatrix}, \\ (\mathbb{F}\,u' + \mathbb{G}\,v')' = (x'(t)\cdot\sigma_v)' &= \frac{1}{2}\begin{pmatrix} u' & v' \end{pmatrix}\begin{pmatrix} \mathbb{E} & \mathbb{F} \\ \mathbb{F} & \mathbb{G} \end{pmatrix}_v \begin{pmatrix} u' \\ v' \end{pmatrix}.\end{aligned} \tag{32}$$

Alternative formulation.

$$u'' + \begin{pmatrix} u' & v' \end{pmatrix}\begin{pmatrix} \Gamma_{11}^1 & \Gamma_{12}^1 \\ \Gamma_{12}^1 & \Gamma_{22}^1 \end{pmatrix}\begin{pmatrix} u' \\ v' \end{pmatrix} = 0, \tag{33}$$

and

$$v'' + \begin{pmatrix} u' & v' \end{pmatrix}\begin{pmatrix} \Gamma_{11}^2 & \Gamma_{12}^2 \\ \Gamma_{12}^2 & \Gamma_{22}^2 \end{pmatrix}\begin{pmatrix} u' \\ v' \end{pmatrix} = 0. \tag{34}$$

- Christoffel symbols.

$$\begin{aligned}\sigma_{uu} &= \Gamma_{11}^1\,\sigma_u + \Gamma_{11}^2\,\sigma_v + \mathbb{L}\,N, \\ \sigma_{uv} &= \Gamma_{12}^1\,\sigma_u + \Gamma_{12}^2\,\sigma_v + \mathbb{M}\,N, \\ \sigma_{vv} &= \Gamma_{22}^1\,\sigma_u + \Gamma_{22}^2\,\sigma_v + \mathbb{N}\,N.\end{aligned} \tag{35}$$

- Make sure you know how to solve (35) to obtain

$$\begin{aligned}\Gamma_{11}^1 &= \frac{\mathbb{G}\,\mathbb{E}_u - 2\,\mathbb{F}\,\mathbb{F}_u + \mathbb{F}\,\mathbb{E}_v}{2\,(\mathbb{E}\,\mathbb{G} - \mathbb{F}^2)}, & \Gamma_{11}^2 &= \frac{2\,\mathbb{E}\,\mathbb{F}_u - \mathbb{E}\,\mathbb{E}_v + \mathbb{F}\,\mathbb{E}_u}{2\,(\mathbb{E}\,\mathbb{G} - \mathbb{F}^2)}, \\ \Gamma_{12}^1 &= \frac{\mathbb{G}\,\mathbb{E}_v - \mathbb{F}\,\mathbb{G}_u}{2\,(\mathbb{E}\,\mathbb{G} - \mathbb{F}^2)}, & \Gamma_{12}^2 &= \frac{\mathbb{E}\,\mathbb{G}_u - \mathbb{F}\,\mathbb{E}_v}{2\,(\mathbb{E}\,\mathbb{G} - \mathbb{F}^2)}, \\ \Gamma_{22}^1 &= \frac{2\,\mathbb{G}\,\mathbb{F}_v - \mathbb{G}\,\mathbb{G}_u - \mathbb{F}\,\mathbb{G}_v}{2\,(\mathbb{E}\,\mathbb{G} - \mathbb{F}^2)}, & \Gamma_{22}^2 &= \frac{\mathbb{E}\,\mathbb{G}_v - 2\,\mathbb{F}\,\mathbb{F}_v + \mathbb{F}\,\mathbb{G}_u}{2\,(\mathbb{E}\,\mathbb{G} - \mathbb{F}^2)}.\end{aligned} \tag{36}$$

2.4. Parallel transport

- Covariant derivative.

$$\nabla_\gamma w = w' - (w'\cdot N)\,N. \tag{37}$$

Projection of w' onto T_pS.

- $w(t) = \alpha(t)\,\sigma_u + \beta(t)\,\sigma_v$ is parallel along $x(t)$:

$$\alpha' + \left[\begin{pmatrix} \Gamma_{11}^1 & \Gamma_{12}^1 \\ \Gamma_{12}^1 & \Gamma_{22}^1 \end{pmatrix}\begin{pmatrix} u' \\ v' \end{pmatrix}\right]\cdot\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = 0, \tag{38}$$

$$\beta' + \left[\begin{pmatrix} \Gamma_{11}^2 & \Gamma_{12}^2 \\ \Gamma_{12}^2 & \Gamma_{22}^2 \end{pmatrix}\begin{pmatrix} u' \\ v' \end{pmatrix}\right]\cdot\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = 0. \tag{39}$$

3. An Example.

Consider the surface patch $\sigma(u,v) := (u\cos v, u\sin v, u)$ defined on $u > 0$, $0 < v < 2\,\pi$. We try to understand as much of this surface as possible.

a) What does this surface look like?

Cone.

b) First fundamental form.

We calculate

$$\sigma_u = (\cos v, \sin v, 1), \qquad \sigma_v = (-u \sin v, u \cos v, 0). \tag{40}$$

Therefore

$$\mathbb{E} = 2, \qquad \mathbb{F} = 0, \qquad \mathbb{G} = u^2. \tag{41}$$

c) Measurements.

- Arc length of the curve $u = 1$, $v \in (0, 2\pi)$.
  1. First parametrize: $u(t) = 1, v(t) = t$.
  2. Calculate

$$\begin{aligned} L &= \int_0^{2\pi} \sqrt{\mathbb{E}\,(u')^2 + 2\,\mathbb{F}\,u'\,v' + \mathbb{G}\,(v')^2}\,\mathrm{d}t \\ &= \int_0^{2\pi} \sqrt{2 \cdot 0^2 + 2 \cdot 0 \cdot 0 \cdot 1 + 1^2 \cdot 1^2}\,\mathrm{d}t = 2\,\pi. \end{aligned} \tag{42}$$

- Angle between $u = 1$ and $v = \pi$.
  1. First parametrize:
     - $x(t)$: $u = 1$: $u(t) = 1, v(t) = t$;
     - $\tilde{x}(\tilde{t})$: $v = \pi$: $\tilde{u}(\tilde{t}) = \tilde{t}, \tilde{v}(\tilde{t}) = \pi$.
  2. Find intersection point.
     - The point is $u = 1, v = \pi$.
     - On $x(t)$ it is at $t_0 = \pi$;
     - On $\tilde{x}(\tilde{t})$ it is at $\tilde{t}_0 = 1$.
  3. Calculate first fundamental form.
     At $u = 1, v = \pi$, we have $\mathbb{E} = 2, \mathbb{F} = 0, \mathbb{G} = 1$.
  4. Calculate the angle.

$$u'(t_0) = 0, \qquad v'(t_0) = 1; \qquad \tilde{u}'(\tilde{t}_0) = 1, \qquad \tilde{v}'(\tilde{t}_0) = 1. \tag{43}$$

$$\cos\theta = 0 \Longrightarrow \theta = \frac{\pi}{2}. \tag{44}$$

- Area for $U = \{0 < u < 1, 0 < v < 2\,\pi\}$.

$$\begin{aligned} A &= \int_U \sqrt{\mathbb{E}\,\mathbb{G} - \mathbb{F}^2}\,\mathrm{d}u\,\mathrm{d}v \\ &= \int_U \sqrt{2}\,|u|\,\mathrm{d}u\,\mathrm{d}v \\ &= \int_0^{2\pi} \left[\int_0^1 \sqrt{2}\,u\,\mathrm{d}u \right] \mathrm{d}v = \sqrt{2}\,\pi. \end{aligned} \tag{45}$$

d) Second fundamental form.

- Calculate $N(u,v)$.

$$N(u,v)=\frac{\sigma_u\times\sigma_v}{\|\sigma_u\times\sigma_v\|}=\frac{(-\cos v,-\sin v,1)}{\sqrt{2}}. \tag{46}$$

- Calculate $\sigma_{uu},\sigma_{uv},\sigma_{vv}$.

$$\sigma_{uu}=(0,0,0);\quad \sigma_{uv}=(-\sin v,\cos v,0);\quad \sigma_{vv}=(-u\cos v,-u\sin v,0). \tag{47}$$

- Therefore

$$\mathbb{L}=0;\qquad \mathbb{M}=0;\qquad \mathbb{N}=\frac{u}{\sqrt{2}}. \tag{48}$$

e) Normal curvature along $w=\alpha\,\sigma_u+\beta\,\sigma_v$.

$$\kappa_n(w)=\frac{\mathbb{L}\,\alpha^2+2\,\mathbb{M}\,\alpha\,\beta+\mathbb{N}\,\beta^2}{\mathbb{E}\,\alpha^2+2\,\mathbb{F}\,\alpha\,\beta+\mathbb{G}\,\beta^2}=\frac{1}{\sqrt{2}}\,\frac{u\,\beta^2}{2\,\alpha^2+u^2\,\beta^2}. \tag{49}$$

f) Principal curvatures and principal directions.

- $\kappa_{1,2}$:

$$\det\begin{pmatrix}\mathbb{L}-\kappa\,\mathbb{E} & \mathbb{M}-\kappa\,\mathbb{F}\\ \mathbb{M}-\kappa\,\mathbb{F} & \mathbb{N}-\kappa\,\mathbb{G}\end{pmatrix}=2\,\kappa\left(u^2\,\kappa-\frac{u}{\sqrt{2}}\right). \tag{50}$$

Therefore

$$\kappa_1=\frac{1}{\sqrt{2}\,u},\qquad \kappa_2=0. \tag{51}$$

- $t_{1,2}$: Solving

$$\left[\begin{pmatrix}\mathbb{L} & \mathbb{M}\\ \mathbb{M} & \mathbb{N}\end{pmatrix}-\kappa_i\begin{pmatrix}\mathbb{E} & \mathbb{F}\\ \mathbb{F} & \mathbb{G}\end{pmatrix}\right]\begin{pmatrix}a_i\\ b_i\end{pmatrix}=0 \tag{52}$$

we obtain

$$\begin{pmatrix}a_1\\ b_1\end{pmatrix}\parallel\begin{pmatrix}0\\ 1\end{pmatrix},\qquad \begin{pmatrix}a_2\\ b_2\end{pmatrix}\parallel\begin{pmatrix}1\\ 0\end{pmatrix}. \tag{53}$$

Thus

$$t_1\parallel\sigma_v,\qquad t_2\parallel\sigma_u. \tag{54}$$

We can solve

$$t_1=(-\sin v,\cos v,0),\qquad t_2=\frac{1}{\sqrt{2}}\,(\cos v,\sin v,1). \tag{55}$$

g) Mean and Gaussian curvatures.

- Mean curvature.

$$H=\frac{\kappa_1+\kappa_2}{2}=\frac{1}{2\,\sqrt{2}\,u}. \tag{56}$$

- Gaussian curvature.

$$K=\kappa_1\,\kappa_2=0. \tag{57}$$

h) Christoffel symbols. Recall

$$\begin{aligned}\sigma_{uu} &= \Gamma_{11}^1\,\sigma_u + \Gamma_{11}^2\,\sigma_v + \mathbb{L}\,N,\\ \sigma_{uv} &= \Gamma_{12}^1\,\sigma_u + \Gamma_{12}^2\,\sigma_v + \mathbb{M}\,N,\\ \sigma_{vv} &= \Gamma_{22}^1\,\sigma_u + \Gamma_{22}^2\,\sigma_v + \mathbb{N}\,N.\end{aligned} \tag{58}$$

- $\Gamma_{11}^1, \Gamma_{11}^2$. We have

$$\begin{aligned}0 = \sigma_{uu}\cdot\sigma_u &= \mathbb{E}\,\Gamma_{11}^1 + \mathbb{F}\,\Gamma_{11}^2 = 2\,\Gamma_{11}^1,\\ 0 = \sigma_{uu}\cdot\sigma_v &= \mathbb{F}\,\Gamma_{11}^1 + \mathbb{G}\,\Gamma_{11}^2 = u^2\,\Gamma_{11}^2.\end{aligned} \tag{59}$$

  Thus $\Gamma_{11}^1 = \Gamma_{11}^2 = 0$.

- Similarly, solve the other four.

$$\Gamma_{12}^1 = 0, \qquad \Gamma_{12}^2 = \frac{1}{u}, \qquad \Gamma_{22}^1 = -\frac{u}{2}, \qquad \Gamma_{22}^2 = 0. \tag{60}$$

i) Geodesics. Recall the geodesics equations:

$$u'' + \begin{pmatrix} u' & v' \end{pmatrix}\begin{pmatrix} \Gamma_{11}^1 & \Gamma_{12}^1\\ \Gamma_{12}^1 & \Gamma_{22}^1 \end{pmatrix}\begin{pmatrix} u'\\ v' \end{pmatrix} = 0, \tag{61}$$

and

$$v'' + \begin{pmatrix} u' & v' \end{pmatrix}\begin{pmatrix} \Gamma_{11}^2 & \Gamma_{12}^2\\ \Gamma_{12}^2 & \Gamma_{22}^2 \end{pmatrix}\begin{pmatrix} u'\\ v' \end{pmatrix} = 0. \tag{62}$$

They become

$$u'' - \frac{u}{2}\,(v')^2 = 0, \qquad v'' + \frac{2\,u'\,v'}{u} = 0. \tag{63}$$

We see that $v = \text{constant}$ are geodesics but $u = \text{constant}$ are not.

Exercise 1. If we substitute $u = v = t$ into (63), we see that the equations are not satisfied. Does this mean $u = v$ is not a geodesic? If not, how should we check whether $u = v$ is a geodesic?