Midterm 1

(Sept. 29, 2016, 11am–12pm. Total 15+3 pts)

NAME: ID#:

- There are four problems (total 15 pts + 3 bonus pts).
- Please write clearly and show enough work.
- The following formulas may be useful:
  - Curvature for general parametrization:

$$\kappa(t) = \frac{\|x'(t) \times x''(t)\|}{\|x'(t)\|^3}. \tag{1}$$

  - Torsion for arc length parametrization:

$$\tau(s) = \frac{(x'(s) \times x''(s)) \cdot x'''(s)}{\|x''(s)\|^2} \tag{2}$$

  - Torsion for general parametrization:

$$\tau(t) = \frac{(x'(t) \times x''(t)) \cdot x'''(t)}{\|x'(t) \times x''(t)\|^2} \tag{3}$$

QUESTION 1. (5 PTS) *Rewrite the curve* $x(t) = (\sin t, \cos t, 3\,t)$ *in arc length parametrization.*

Solution. We calculate

$$x'(t) = (\cos t, -\sin t, 3) \Longrightarrow \|x'(t)\| = \sqrt{10}. \tag{4}$$

Next we solve $S'(t) = \sqrt{10}$ to obtain $S(t) = \sqrt{10}\,t$ whose inverse function is $T(s) = \frac{s}{\sqrt{10}}$. Therefore the arc length parametrization is

$$x(s) = \left(\sin\frac{s}{\sqrt{10}}, \cos\frac{s}{\sqrt{10}}, \frac{3}{\sqrt{10}}\,s\right). \tag{5}$$

We easily check that $\|x'(s)\| = 1$ now.

QUESTION 2. (5 PTS) *Let* $x(t) = (-\sin t, 1 - \cos t, t)$. *Calculate the unit tangent vector* $T(t)$*, the unit normal vector* $N(t)$*, the unit binormal vector* $B(t)$*, the curvature* $\kappa(t)$*, and the torsion* $\tau(t)$*.*

Solution. We calculate

$$x'(t) = (-\cos t, \sin t, 1), \quad x''(t) = (\sin t, \cos t, 0), \quad x'''(t) = (\cos t, -\sin t, 0), \tag{6}$$

$$x'(t) \times x''(t) = (-\cos t, \sin t, -1), \tag{7}$$

$$(x'(t) \times x''(t)) \cdot x'''(t) = -1, \tag{8}$$

$$\|x'(t)\| = \sqrt{2}, \qquad \|x'(t) \times x''(t)\| = \sqrt{2}. \tag{9}$$

Therefore

$$T(t) = \frac{x'(t)}{\|x'(t)\|} = \frac{(-\cos t, \sin t, 1)}{\sqrt{2}}, \tag{10}$$

$$B(t) = \frac{x'(t) \times x''(t)}{\|x'(t) \times x''(t)\|} = \frac{(-\cos t, \sin t, -1)}{\sqrt{2}}, \tag{11}$$

$$N(t) = B(t) \times T(t) = -(\sin t, \cos t, 0), \tag{12}$$

$$\kappa(t) = \frac{\|x'(t) \times x''(t)\|}{\|x'(t)\|^3} = \frac{1}{2}, \tag{13}$$

$$\tau(t) = \frac{(x'(t) \times x''(t)) \cdot x'''(t)}{\|x'(t) \times x''(t)\|^2} = -\frac{1}{2}. \tag{14}$$

QUESTION 3. (5 PTS + 1 BONUS PT) *Consider the surface* S *given by* $x_3 = x_1^2 + x_2^2$. *Let* $N(p)$: $S \mapsto \mathbb{S}^2$ *be the unit* ***upward*** *normal of* S *at* $p \in S$. *Here* $\mathbb{S}^2$ *is the unit sphere* $x_1^2 + x_2^2 + x_3^2 = 1$. *Let* $p_0 = (1, 1, 2)$.

a) (1 PT) *Calculate* $N(p_0)$.

b) (3 PTS) *Compute the matrix representation of the differential map* $D_{p_0} N$.

c) (1 PT) *Prove that* $v = (-1, 1, 0) \in T_{p_0} S$

d) (BONUS 1 PT) *Calculate* $D_{p_0} N(v)$.

Solution.

a) We take $\sigma: \mathbb{R}^2 \mapsto S$ to be $\sigma(u, v) = (u, v, u^2 + v^2)$. Then $p_0 = \sigma(1, 1)$, and

$$\sigma_u = (1, 0, 2u), \qquad \sigma_v = (0, 1, 2v), \tag{15}$$

$$\sigma_u \times \sigma_v = (-2u, -2v, 1). \tag{16}$$

We see that it points upward. Therefore $N(\sigma(u, v)) = \frac{(-2u, -2v, 1)}{\sqrt{1 + 4u^2 + 4v^2}}$ and $N(p_0) = \frac{1}{3}(-2, -2, 1)$.

b) Since N points upward, we take $\tilde{\sigma}: \{(u, v) | \, u^2 + v^2 < 1\} \mapsto S$ to be the surface patch for the upper hemisphere: $\tilde{\sigma}(\tilde{u}, \tilde{v}) = \left(\tilde{u}, \tilde{v}, \sqrt{1 - \tilde{u}^2 - \tilde{v}^2}\right)$. This gives

$$\tilde{\sigma}^{-1}(x, y, z) = (x, y). \tag{17}$$

Consequently

$$F(u, v) := \tilde{\sigma}^{-1}(N(\sigma(u, v))) = \frac{(-2u, -2v)}{\sqrt{1 + 4u^2 + 4v^2}}. \tag{18}$$

The Jacobian is

$$\begin{pmatrix} \frac{-2}{\sqrt{1+4u^2+4v^2}}+\frac{8u^2}{(\sqrt{1+4u^2+4v^2})^3} & \frac{8uv}{(\sqrt{1+4u^2+4v^2})^3} \\ \frac{8uv}{(\sqrt{1+4u^2+4v^2})^3} & \frac{-2}{\sqrt{1+4u^2+4v^2}}+\frac{8v^2}{(\sqrt{1+4u^2+4v^2})^3} \end{pmatrix} \tag{19}$$

which gives

$$DF(1,1)=\begin{pmatrix} -\frac{10}{27} & \frac{8}{27} \\ \frac{8}{27} & \frac{-10}{27} \end{pmatrix}. \tag{20}$$

c) As $v \perp N(p_0)$ we see that $v \in T_{p_0}S$.

d) We calculate

$$\sigma_u(1,1)=(1,0,2), \qquad \sigma_v(1,1)=(0,1,2). \tag{21}$$

Thus

$$v=-\sigma_u+\sigma_v. \tag{22}$$

Next we note that $N(p_0)=\tilde{\sigma}\left(-\frac{2}{3},-\frac{2}{3}\right)$ and calculate

$$\tilde{\sigma}_{\tilde{u}}=\left(1,0,\frac{-\tilde{u}}{\sqrt{1-\tilde{u}^2-\tilde{v}^2}}\right), \qquad \tilde{\sigma}_{\tilde{v}}=\left(0,1,\frac{-\tilde{v}}{\sqrt{1-\tilde{u}^2-\tilde{v}^2}}\right) \tag{23}$$

which gives

$$\tilde{\sigma}_{\tilde{u}}\left(-\frac{2}{3},-\frac{2}{3}\right)=(1,0,2), \qquad \tilde{\sigma}_{\tilde{v}}\left(-\frac{2}{3},-\frac{2}{3}\right)=(0,1,2). \tag{24}$$

Consequently we have

$$\begin{pmatrix} \tilde{a} \\ \tilde{b} \end{pmatrix}=\begin{pmatrix} -\frac{10}{27} & \frac{8}{27} \\ \frac{8}{27} & \frac{-10}{27} \end{pmatrix}\begin{pmatrix} -1 \\ 1 \end{pmatrix}=\begin{pmatrix} \frac{2}{3} \\ -\frac{2}{3} \end{pmatrix}. \tag{25}$$

Therefore

$$D_{p_0}N(v)=\frac{2}{3}\,[\tilde{\sigma}_{\tilde{u}}-\tilde{\sigma}_{\tilde{v}}]=\left(\frac{2}{3},-\frac{2}{3},0\right). \tag{26}$$

QUESTION 4. (BONUS. 2 PTS) *Let $x(s)$ be arc length parametrized. Let $T(s)$ and $B(s)$ be its unit tangent and binormal vectors. Let $\kappa(s), \tau(s)$ be its curvature and torsion. Prove that*

$$[(T(s)\times T'(s))\cdot T''(s)]\,[(B(s)\times B'(s))\cdot B''(s)] = \kappa(s)^3\,\tau(s)^3. \tag{27}$$

Proof. We have

$$T'(s) = \kappa\,N(s), \quad T''(s) = \kappa'\,N + \kappa\,N' = -\kappa^2\,T + \kappa'\,N + \kappa\,\tau\,B, \tag{28}$$

$$B' = -\tau N, \quad B'' = -\tau'\,N - \tau\,N' = \tau\,\kappa\,T - \tau'\,N - \tau^2\,B. \tag{29}$$

Therefore

$$(T\times T')\cdot T'' = \kappa^2\,\tau, \quad (B\times B')\cdot B'' = \kappa\,\tau^2 \tag{30}$$

and the conclusion follows. □