Review for Midterm 1

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1. Curves

1.1. Concepts and theorems

- PARAMETRIZED CURVE. A parametrized curve in $\mathbb{R}^n$ is a map[1] $x\colon (\alpha, \beta) \mapsto \mathbb{R}^n$, for some α, β with $-\infty \leqslant \alpha < \beta \leqslant \infty$.
- ARC LENGTH PARAMETRIZATION. A parametrization $x(t)$ such that $\|x'(t)\| = 1$ everywhere.
- THEOREMS.
  - $\kappa(t) = 0$ for all $t \Longleftrightarrow$ the curve is part of a straight line.
  - $\tau(t) = 0$ and $\kappa(t) \neq 0$ for all $t \Longrightarrow$ the curve is a plane curve.

    Exercise 1. Does $\Longleftarrow$ hold?
  - $\tau(t) = 0$ and $\kappa(t) = \kappa_0$ constant for all $t \Longleftrightarrow$ the curve is part of a circle.

1.2. Formulas

- ARC LENGTH.

$$L = \int_a^b \|x'(t)\| \, \mathrm{d}t. \tag{1}$$

- FINDING ARC LENGTH PARAMETRIZATION.

  Given $x(t)$. To find its arc length parametrization,

  i. Find $S(t)$ such that $S'(t) = \|x'(t)\|$.

  ii. Find the inverse function $T(s)$. That is solve $S(T(s)) = s$.

  iii. The arc length parametrization is given by $x(T(s))$.

- GEOMETRIC QUANTITIES.

	Arc length parametrization	General parametrization
Unit tangent vector T	$x'(s)$	$\frac{x'}{\|x'\|}$
Unit normal vector N	$\frac{x''(s)}{\|x''(s)\|}$	$B \times T$
Unit binormal vector B	$T(s) \times N(s) = \frac{x'(s) \times x''(s)}{\|x''(s)\|}$	$\frac{x' \times x''}{\|x' \times x''\|}$
Curvature κ	$\|x''(s)\|$	$\frac{\|x' \times x''\|}{\|x'\|^3}$
Torsion τ	$\frac{(x'(s) \times x''(s)) \cdot x'''(s)}{\|x''(s)\|^2}$	$\frac{(x' \times x'') \cdot x'''}{\|x' \times x''\|^2}$

Table 1. Geometric quantities

Warning. Only the formulas in red will be provided in exams.

- THE FRENET-SERRET EQUATIONS.

$$\begin{aligned} T' &= \kappa N \\ N' &= -\kappa T + \tau B \\ B' &= -\tau N \end{aligned} . \tag{2}$$

1. Another name for "function".

Warning. (2) only holds for arc length parametrization.

1.3. Examples

Example 1. Consider the curve $(e^t \cos t, e^t \sin t, e^t)$, $t \in \mathbb{R}$.

- Arc length. We calculate

$$x'(t) = (e^t \cos t - e^t \sin t, e^t \sin t + e^t \cos t, e^t) \tag{3}$$

therefore

$$\|x'(t)\| = \sqrt{3}\, e^t \tag{4}$$

Consequently the arc length from $x(a)$ to $x(b)$ is given by

$$L = \int_a^b \sqrt{3}\, e^t = \sqrt{3}\,(e^b - e^a). \tag{5}$$

- Arc length parametrization. We need $s = S(t)$ such that $S'(t) = \sqrt{3}\, e^t$. This gives $S(t) = \sqrt{3}\, e^t$ and the inverse function is $T(s) = \ln\!\left(s/\sqrt{3}\right)$. Therefore the arc length parametrization is given by $\left(\frac{s}{\sqrt{3}} \cos\!\left(\ln\!\left(s/\sqrt{3}\right)\right), \frac{s}{\sqrt{3}} \sin\!\left(\ln\!\left(s/\sqrt{3}\right)\right), \frac{s}{\sqrt{3}}\right)$.

- T, N, B, κ, τ (Frenet apparatus).

  We calculate

$$x''(t) = (-2\, e^t \sin t, 2\, e^t \cos t, e^t), \tag{6}$$

$$x'''(t) = (-2\, e^t \cos t - 2\, e^t \sin t, 2\, e^t \cos t - 2\, e^t \sin t, e^t), \tag{7}$$

and

$$x'(t) \times x''(t) = e^{2t}\,(\sin t - \cos t, -\cos t - \sin t, 2), \tag{8}$$

which leads to

$$\|x'(t) \times x''(t)\| = \sqrt{6}\, e^{2t}, \tag{9}$$

and

$$(x'(t) \times x''(t)) \cdot x'''(t) = 2\, e^{3t}. \tag{10}$$

Consequently we have

$$T(t) = \frac{1}{\sqrt{3}}\,(\cos t - \sin t, \sin t + \cos t, 1), \tag{11}$$

$$B(t) = \frac{1}{\sqrt{6}}\,(\sin t - \cos t, -\cos t - \sin t, 2), \tag{12}$$

$$N(t) = \frac{1}{\sqrt{2}}\,(-\cos t - \sin t, \cos t - \sin t, 0), \tag{13}$$

$$\kappa(t) = \frac{\sqrt{2}}{3}\, e^{-t}, \tag{14}$$

$$\tau(t) = \frac{1}{3}\, e^{-t}. \tag{15}$$

Example 2. Let $x(s)$ be a regular arclength-parametrized curve with nonzero curvature. The normal line to x at $x(s)$ is the line through $x(s)$ with direction vector $N(s)$. Suppose all the normal lines to x pass through a fixed point. What can you say about the curve?

Solution. We take the fixed point to be the origin. Then we have $x(s) \parallel N(s)$. This means $x(s) \cdot T(s) = 0$ and consequently $x(s) \cdot x'(s) = 0$ so $x(s)$ lies in a sphere. On the other hand $x(s) \parallel x''(s)$ implies the existence of a scalar function $\lambda(s)$ such that $x(s) = \lambda(s)\, x''(s)$. Taking derivative we have $x'(s) = \lambda'(s)\, x''(s) + \lambda(s)\, x'''(s)$ so $(x'(s) \times x''(s)) \cdot x'''(s) = 0$ and therefore $\tau(s) = 0$. From this we conclude that $x(s)$ is a plane curve. Thus $x(s)$ is a circle.

2. Surfaces

2.1. Concepts

- Surface patch. A surface patch $\sigma: U \mapsto \mathbb{R}^3$ is called regular if it is smooth and the vectors σ_u and σ_v are linearly independent at all points $(u, v) \in U$.
- Surface. $S \subset \mathbb{R}^3$ such that at every $p \in S$ there is a surface patch covering it.

2.2. Formulas

- Unit normal vector.

$$\pm \frac{\sigma_u \times \sigma_v}{\|\sigma_u \times \sigma_v\|}. \tag{16}$$

- Tangent plane.

$$T_p S = \{a\, \sigma_u(u_0, v_0) + b\, \sigma_v(u_0, v_0) \colon a, b \in \mathbb{R}\}. \tag{17}$$

- Surface area.

$$\int_U \|\sigma_u \times \sigma_v\|\, \mathrm{d}u \mathrm{d}v \tag{18}$$

- Differential of function $f: S \mapsto \tilde{S}$. Four steps.
    i. Take a surface patch $\sigma: U \mapsto S$ covering p: $\sigma(u_0, v_0) = p$.
    ii. Take a surface patch $\tilde{\sigma}: \tilde{U} \mapsto \tilde{S}$ covering $f(p)$.
    iii. Compute $F := (\tilde{\sigma})^{-1} \circ f \circ \sigma: U \mapsto \tilde{U}$.
    iv. Calculate the Jacobian DF.

2.3. Examples

We will go through Questions 3–5 in homework 2.