Lectures 20–21: Surfaces and Curves in $\mathbb{R}^n$

Disclaimer. As we have a textbook, this lecture note is for guidance and supplement only. It should not be relied on when preparing for exams.

In this lecture we review what we have learned and try to generalize to obtain a theory for m-dimensional surfaces in $\mathbb{R}^n$.

The material is optional.

I try my best to make the examples in this note different from examples in the textbook. Please read the textbook carefully and try your hands on the exercises. During this please don't hesitate to contact me if you have any questions.

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- In this lecture we use notation $f_{,i} := \frac{\partial f}{\partial u_i}$, $f_{,ij} := \frac{\partial^2 f}{\partial u_i \partial u_j}$, etc.

1. $n = m + 1$.

- Surface patch. Naturally we represent an m-dimensional surface patch in $\mathbb{R}^n$ as

$$\sigma: U \mapsto \mathbb{R}^n, \qquad \sigma(u_1, \ldots, u_m) = (\sigma_1(u_1, \ldots, u_m), \ldots, \sigma_n(u_1, \ldots, u_m)). \tag{1}$$

- Tangent and normal vectors.

  The tangent plane is

$$T_p S = \operatorname{span}\{\sigma_{,1}, \ldots, \sigma_{,m}\} \tag{2}$$

  which can be identified as $\mathbb{R}^m$.

  Since $n = m + 1$, there are exactly two unit normal vectors. We pick one and called it the unit normal vector and denote it by N.

- First fundamental form, measurement.

  Define

$$g_{ij} := \sigma_{,i} \cdot \sigma_{,j}, \qquad i, j = 1, 2, \ldots, m. \tag{3}$$

  We call (g_{ij}) the metric tensor. We also use (g_{ij}) to denote the $m \times m$ matrix whose (i, j) entry is g_{ij} for every $1 \leqslant i, j \leqslant m$.

  Then we can easily have, for vectors $w = \sum_{i=1}^m w_i \sigma_{,i}$, $\tilde{w} = \sum_{j=1}^m \tilde{w}_j \sigma_{,j}$,

$$\|w\| = \sqrt{\sum_{i,j=1}^m g_{ij} w_i w_j}, \tag{4}$$

$$\cos \angle(w, \tilde{w}) = \frac{\sum_{i,j=1}^m g_{ij} w_i \tilde{w}_j}{\|w\| \, \|\tilde{w}\|}. \tag{5}$$

  The first fundamental form is then

$$I = \sum_{i,j=1}^m g_{ij} \, \mathrm{d}u_i \, \mathrm{d}u_j. \tag{6}$$

  Also the volume of $\sigma(U)$ is

$$\int_U \sqrt{\det(g_{ij})} \, \mathrm{d}u_1 \cdots \mathrm{d}u_m. \tag{7}$$

- Second fundamental form.

  We denote

$$b_{ij} := \sigma_{,ij} \cdot N = b_{ji}. \tag{8}$$

  Then the second fundamental form is

$$\sum_{i,j=1}^m b_{ij} \, \mathrm{d}u_i \, \mathrm{d}u_j. \tag{9}$$

  Note that by definition (b_{ij}) is symmetric.

- Gauss map, Weingarten map.

We define the Gauss map $\mathcal{G}: U \mapsto \mathbb{S}^m$ through $\mathcal{G}(\sigma(u_1, \ldots, u_m)) = N(u_1, \ldots, u_m)$. The corresponding Weingarten map $\mathcal{W} := -D\,\mathcal{G}$ is then characterized by

$$\mathcal{W}\left(\sum_{i=1}^m w_i\,\sigma_{,i}\right) = \sum_{i=1}^m w_i\,(-N_{,i}). \tag{10}$$

Now notice that there hold

$$b_{ij} = \sigma_{,ij} \cdot N = -\sigma_{,i} \cdot N_{,j} = -\sigma_{,j} \cdot N_{,i} \tag{11}$$

Thus if we write

$$-N_{,i} = \sum_{k=1}^m a_{ik}\,\sigma_{,k}, \tag{12}$$

there would hold

$$b_{ij} = g_{jk}\,a_{ik} \Longrightarrow (b_{ij}) = (g_{ij})\,(a_{ij})^T \tag{13}$$

and consequently we have the matrix relation

$$(a_{ij})^T = (g_{ij})^{-1}\,(b_{ij}). \tag{14}$$

- Curvatures.

Let $\kappa_1, \ldots, \kappa_m$ be the eigenvalues of the Weingarten map. Then they solve

$$\det((a_{ij})^T - \kappa\,I) = 0 \Longleftrightarrow \det[(b_{ij}) - \kappa\,(g_{ij})] = 0. \tag{15}$$

We can call κ_1, …, κ_m "principal curvatures", and define the mean and Gaussian curvatures as

$$H := \frac{\kappa_1 + \cdots + \kappa_m}{m}, \qquad K := \kappa_1 \cdots \kappa_m. \tag{16}$$

We easily see that

$$H = \mathrm{tr}[(g_{ij})^{-1}\,(b_{ij})], \qquad K = \frac{\det(b_{ij})}{\det(g_{ij})}. \tag{17}$$

Of course, the eigenvectores corresponding to each κ_j are the "principal vectors". If we have a coordinate system that is parallel to these "principal vectors", then both (g_{ij}) and (b_{ij}) are diagonal.

Remark 1. It is immediate that $K = \lim_{\Omega \subset S, \Omega \longrightarrow \{p\}} \frac{\mathrm{Vol}(\mathcal{G}(\Omega))}{\mathrm{Vol}(\Omega)}$.

- Christoffel symbols.

Write

$$\sigma_{,ij} = \sum_{l=1}^m \Gamma_{ij}^l\,\sigma_{,l} + b_{ij}\,N. \tag{18}$$

Multiply both sides by $\sigma_{,k}$ we see that

$$\begin{aligned}\sum_{l=1}^m \Gamma_{ij}^l\,g_{lk} &= \sigma_{,ij} \cdot \sigma_{,k} = (g_{ik})_{,j} - \sigma_{,jk} \cdot \sigma_{,i} \\ &= (g_{ik})_{,j} - \left[\sum_{l=1}^m \Gamma_{jk}^l\,\sigma_{,l}\right] \cdot \sigma_{,i} \\ &= (g_{ik})_{,j} - \sum_{l=1}^m \Gamma_{jk}^l\,g_{li}. \end{aligned} \tag{19}$$

Therefore (using $,j$ to denote the u^j derivative)

$$g_{ik,j} = \sum_{l=1}^{m} g_{lk}\,\Gamma^l_{ij} + \sum_{l=1}^{m} g_{li}\,\Gamma^l_{jk}. \tag{20}$$

Permuting i, j, k we see that

$$g_{kj,i} = \sum_{l=1}^{m} g_{lj}\,\Gamma^l_{ki} + \sum_{l=1}^{m} g_{lk}\,\Gamma^l_{ij}, \tag{21}$$

$$g_{ji,k} = \sum_{l=1}^{m} g_{li}\,\Gamma^l_{jk} + \sum_{l=1}^{m} g_{lj}\,\Gamma^l_{ki}. \tag{22}$$

Note that the terms with same color coincide. Thus we have

$$\sum_{l=1}^{m} g_{lk}\,\Gamma^l_{ij} = \frac{1}{2}\,[g_{ik,j} + g_{jk,i} - g_{ij,k}] \tag{23}$$

or

$$\Gamma^l_{ij} = \frac{1}{2}\sum_{k=1}^{m} (g_{ij})^{-1}_{lk}\,[g_{ik,j} + g_{jk,i} - g_{ij,k}]. \tag{24}$$

- Covariant derivative, parallel transport, geodesics.

  Again we define

$$\nabla_\gamma w := \text{Projection of } w' \text{ onto } T_pS. \tag{25}$$

Consider the curve

$$x(s) := \sigma(u_1(s), \ldots, u_m(s)). \tag{26}$$

Let $w(s) = w_1(s)\,\sigma_{,1} + \cdots + w_m(s)\,\sigma_{,m}$ be a tangent vector field. Then we have

$$\begin{aligned}
\nabla_\gamma w(s) &= w'(s) - (w'(s)\cdot N)\,N \\
&= \sum_{i=1}^{m} w_i'\,\sigma_{,i} + \sum_{i,j=1}^{m} w_i\,u_j'\,[\sigma_{,ij} - (\sigma_{,ij}\cdot N)\,N] \\
&= \sum_{i=1}^{m} w_i'\,\sigma_{,i} + \sum_{i,j=1}^{m} w_i\,u_j' \sum_{k=1}^{m} \Gamma^k_{ij}\,\sigma_{,k} \\
&= \sum_{k=1}^{m} \left[w_k' + \sum_{i,j=1}^{m} \Gamma^k_{ij}\,w_i\,u_j' \right] \sigma_{,k}.
\end{aligned} \tag{27}$$

Thus the parallel transport equation reads

$$w_k' + \sum_{i,j=1}^{m} \Gamma^k_{ij}\,w_i\,u_j' = 0, \qquad k = 1, 2, \ldots, m, \tag{28}$$

and the geodesic equation reads

$$u_k'' + \sum_{i,j=1}^{m} \Gamma^k_{ij}\,u_i'\,u_j' = 0, \qquad k = 1, 2, \ldots, m. \tag{29}$$

Remark 2. We see that (20) actually says $\nabla_\gamma g = 0$, or equivalently,

$$\frac{\mathrm{d}}{\mathrm{d}s}\langle w, \tilde{w}\rangle = \langle \nabla_\gamma w, \tilde{w}\rangle + \langle w, \nabla_\gamma \tilde{w}\rangle \tag{30}$$

where $\langle\cdot,\cdot\rangle$ is the first fundamental form.

- Codazzi and Gauss equations.

  Recalling (18):

$$\sigma_{,ij} = \sum_{l=1}^m \Gamma_{ij}^l \sigma_{,l} + b_{ij} N, \tag{31}$$

we also have

$$\sigma_{,jk} = \sum_{l=1}^m \Gamma_{jk}^l \sigma_{,l} + b_{jk} N, \tag{32}$$

$$\sigma_{,ki} = \sum_{l=1}^m \Gamma_{ki}^l \sigma_{,l} + b_{ki} N. \tag{33}$$

Differentiating the three equations with ∂_k, ∂_i, ∂_j respectively, and using (18), we arrive at

$$\sigma_{,ijk} = \sum_{l=1}^m \left\{\Gamma_{ij,k}^l + \sum_{s=1}^m \Gamma_{ij}^s \Gamma_{sk}^l - b_{ij} a_{kl}\right\} \sigma_{,l} + \left\{b_{ij,k} + \sum_{l=1}^m \Gamma_{ij}^l b_{lk}\right\} N, \tag{34}$$

$$\sigma_{,jki} = \sum_{l=1}^m \left\{\Gamma_{jk,i}^l + \sum_{s=1}^m \Gamma_{jk}^s \Gamma_{si}^l - b_{jk} a_{il}\right\} \sigma_{,l} + \left\{b_{jk,i} + \sum_{l=1}^m \Gamma_{jk}^l b_{li}\right\} N, \tag{35}$$

$$\sigma_{,kij} = \sum_{l=1}^m \left\{\Gamma_{ki,j}^l + \sum_{s=1}^m \Gamma_{ki}^s \Gamma_{sj}^l - b_{ki} a_{jl}\right\} \sigma_{,l} + \left\{b_{ki,j} + \sum_{l=1}^m \Gamma_{ki}^l b_{lj}\right\} N. \tag{36}$$

As the mixed derivatives are equal, we have

  - Codazzi-Mainradi equations

$$b_{ij,k} - b_{jk,i} = \sum_{l=1}^m [\Gamma_{jk}^l b_{li} - \Gamma_{ij}^l b_{lk}]. \tag{37}$$

  - Gauss equations

$$b_{jk} a_{il} - b_{ij} a_{kl} = \Gamma_{jk,i}^l - \Gamma_{ij,k}^l + \sum_{s=1}^m [\Gamma_{jk}^s \Gamma_{si}^l - \Gamma_{ij}^s \Gamma_{sk}^l]. \tag{38}$$

  We denote

$$R_{ijk}^l := \Gamma_{jk,i}^l - \Gamma_{ij,k}^l + \sum_{s=1}^m [\Gamma_{jk}^s \Gamma_{si}^l - \Gamma_{ij}^s \Gamma_{sk}^l]. \tag{39}$$

  and

$$R_{sijk} := \sum_{l=1}^m g_{sl} R_{ijk}^l. \tag{40}$$

Then there holds

$$R_{sijk} = \sum_{l=1}^{m} g_{sl} b_{jk} a_{il} - \sum_{l=1}^{m} g_{sl} b_{ij} a_{kl} = b_{si} b_{kl} - b_{sk} b_{ij}. \tag{41}$$

- Gauss' Remarkable Theorem.

  It is clear that R_{ijk}^l and R_{sijk} are invariant under local isometries. Such invariance also holds for

$$a_{ks} a_{il} - a_{is} a_{kl} = \sum_{j=1}^{m} (g)_{sj}^{-1} R_{ijk}^l. \tag{42}$$

This reduces to the invariance of $K = a_{11} a_{22} - a_{12} a_{12}$ under local isometries when $m = 2$.

On the other hand, the question remains that whether the Gaussian curvature $K = \det(a_{ij})$ is invariant or not. Obviously, if $\det(a_{ij})$ can be determined from all the $a_{ks} a_{il} - a_{is} a_{kl}$ then the answer would be affirmative. This is true when m is even. In fact we have the following formula[1]:

$$K = \frac{1}{2^{m/2} m! \det(g_{ij})} R_{i_1 i_2 j_1 j_2} R_{i_3 i_4 j_3 j_4} \cdots R_{i_{n-1} i_n j_{n-1} j_n} \epsilon^{i_1 \cdots i_n} \epsilon^{j_1 \cdots j_n} \tag{43}$$

where $\epsilon^{i_1 \cdots i_n} = \pm 1$, according to whether $i_1, \ldots, i_n$ is an even or odd permutation. The situation is more complicated when m is odd.

We discuss the two cases now.

  - m is even.

    LEMMA 3. *Let $m \in \mathbb{N}$ be even. Then K is a function of the collection $a_{ks} a_{il} - a_{is} a_{kl}$.*

    Proof. We notice that the collection $a_{ks} a_{il} - a_{is} a_{kl}$ is that of determinants of all 2×2 submatrices of (a_{ij}). Since the determinant of an $n \times n$ matrix can be represented as a sum of products between determinants of its 2×2 submatrices and determinants of its $(n-2) \times (n-2)$ submatrices,[2] it follows that K is a function of $a_{ks} a_{il} - a_{is} a_{kl}$, $i, s, k, l = 1, 2, \ldots, m$. □

  - m is odd.

    - In this case K is not a function of $a_{ks} a_{il} - a_{is} a_{kl}$, as can be seen from the following simple observation: Let $\tilde{A} := -A$. Then $\tilde{a}_{ks} \tilde{a}_{il} - \tilde{a}_{is} \tilde{a}_{kl} = a_{ks} a_{il} - a_{is} a_{kl}$ for all i, s, k, l, but $\det \tilde{A} = -\det A$.
    - On the other hand, we now show that this is the only "freedom" $\det A$ has once determinants of all 2×2 submatrices are fixed. We will prove the following.

1. Prove by C. B. Allendoerfer and W. Fenchel around 1938, and later by S. S. Chern in 1944 for abstract m-dimensional manifolds.

2. Laplace expansion for determinants, see e.g. https://en.wikipedia.org/wiki/Laplace_expansion.

LEMMA 4. *If* $b_{ks}\, b_{il} - b_{is}\, b_{kl} = a_{ks}\, a_{il} - a_{is}\, a_{kl}$ *for all* i, s, k, l*, then* $\det B = \pm \det A$*.*

Proof. We start with the case $m=3$. Denote by C_{ij} the co-factors to the entry a_{ij}, that is

$$C_{ij} = (-1)^{i+j} \det(A \text{ with } i\text{th row and } j\text{th column deleted}). \tag{44}$$

Let $C = (C_{ij})$ be the cofactor matrix. As each C_{ij} is the determinant of a 2×2 submatrix of A, the matrix C is fully determined by $a_{ks}\, a_{il} - a_{is}\, a_{kl}$.

Now note that

$$A^{-1} = \frac{1}{\det A} C^T \Longrightarrow \det C = (\det A)^2. \tag{45}$$

Thus $\det A = \pm \sqrt{\det C}$. Apply the same argument to B we have $\det B = \pm \sqrt{\det C}$ and the conclusion follows.

For general odd m, we notice that the "m is even" result can be applied to each C_{ij}. Therefore

$$A^{-1} = \frac{1}{\det A} C^T \Longrightarrow (\det A)^{m-1} = \det C. \tag{46}$$

As both $\det A$ and $\det C$ are real, we still conclude $\det A = \pm \sqrt{\det C}$. □

– Thus we see that under local isometries, there holds $\tilde{K} = \pm K$. We can conclude $\tilde{K} = K$ if there is a continuous family of local isometries connecting identity and the end isometry. More specifically, let $f: \mathbb{R}^n \mapsto \mathbb{R}^n$ be the local isometry between $\sigma_0(u)$ and $\sigma_1(u)$, if there is a continuous function $F(x,t): \mathbb{R}^{n+1} \mapsto \mathbb{R}^n$ such that

$$F(x,0) = x, \qquad F(x,1) = f(x), \tag{47}$$

and furthermore for every $t_0 \in (0,1)$, $F(x,t_0)$ is a local isometry between $\sigma_0(u)$ and $\sigma_{t_0}(u) := F(\sigma_0(u), t_0)$, then we must have $K_1 = K_0$ thanks to mean value theorem.

QUESTION 5. *Looks like this should always be true at least when the surface patch is small enough. Proof?*

- Gauss-Bonnet.

There is also generalization of Gauss-Bonnet when m is even. This is related to some major contributions of S. S. Chern and reaches pretty far into modern mathematics.[3] Unfortunately I don't know enough to discuss this here and now.

3. https://en.wikipedia.org/wiki/Generalized_Gauss–Bonnet_theorem.

2. $m = 1$.

- Set-up. We consider a curve in $\mathbb{R}^n$:

$$x(t) := (x^1(t), \ldots, x^n(t)), \qquad t \in (\alpha, \beta). \tag{48}$$

- Arc length. The arc length is given by

$$\int_a^b \|x'(t)\| \, \mathrm{d}t. \tag{49}$$

The arc length parametrization $x(s)$ is characterized by $\|x'(s)\| = 1$.

- Tangent, principal normal, curvature. Let $x(s)$ be arc length parametrized. The tangent vector is

$$T(s) = x'(s). \tag{50}$$

The principal normal is then

$$N(s) := \frac{x''(s)}{\|x''(s)\|}. \tag{51}$$

The curvature is then

$$\kappa := \|x''(s)\|. \tag{52}$$

Exercise 1. Let $x(t)$ be a parametrized curve where t is not arc length parameter. Derive the formula for κ. Note that in $\mathbb{R}^n$ for general n there is no "cross-product".

Note that we can also obtain $\kappa(s)$ as measuring "how quickly is $x(s)$ turning away from the tangent line":

$$\kappa = \text{area of the parallelogram spanned by } x'(s), x''(s). \tag{53}$$

For future convenience we denote κ by κ_1, and N by N_1.

- Torsion and more.

We recall that torsion measures how quickly $x(s)$ turns away from the plane spanned by T and N. As a consequence, we have

$$\kappa^2 \tau = \text{volume of the parallelopiped spanned by } x'(s), x''(s), x'''(s). \tag{54}$$

From now on we denote τ by κ_2. We denote by N_2 the unit normal vector in span$\{x', x'', x'''\}$ that is perpendicular to T, N_1 and such that the orthonormal system $\{T, N_1, N_2\}$ is positive.

Exercise 2. Prove that

It is clear that we can go on to define κ_m through

$$\kappa_1^m \kappa_2^{m-1} \cdots \kappa_m := \text{volume of the parallelopiped spanned by } x'(s), \ldots, x^{(m+1)}(s) \tag{55}$$

for $m = 3, 4, \ldots, n-1$, and N_m the unit normal vector in span$\{x', \ldots, x^{(m+1)}\}$ that is perpendicular to $T, N_1, \ldots, N_{m-1}$ such that the orthonormal system $\{T, N_1, \ldots, N_m\}$ is positive.

Thus we have $n-1$ curvatures $\kappa_1 = \kappa, \kappa_2 = \tau, \kappa_3, \ldots, \kappa_{n-1}$.

- Frenet-Serret equations.
  - T'. By definition
$$T' = \kappa_1 N_1. \tag{56}$$
  - N_1'. We differentiate
$$x'''(s) = (\kappa_1 N_1)' = \kappa_1' N_1 + \kappa_1 N_1' \Longrightarrow \kappa_1 N_1' = x'''(s) - \kappa_1' N_1 \in \text{span}\{x', x'', x'''\}. \tag{57}$$
Therefore
$$N_1' = a T + b N_1 + c N_2. \tag{58}$$
As $N_1' \perp N_1$ there holds $b = 0$. On the other hand, from $T \cdot N_1 = 0$ we have
$$\kappa_1 + T \cdot N_1' = 0 \Longrightarrow a = T \cdot N_1' = -\kappa_1 \tag{59}$$
Using (57) again we have
$$x'''(s) = -\kappa_1^2 T + \kappa_1' N_1 + \kappa_1 c N_2 \tag{60}$$
which gives
$$\text{Volume of } \{x', x'', x'''\} = \kappa_1^2 c. \tag{61}$$
Thus $c = \kappa_2$, that is
$$N_1' = -\kappa_1 T + \kappa_2 N_2. \tag{62}$$
  - N_2'. Similarly we have
$$N_2' = a T + b_1 N_1 + b_2 N_2 + b_3 N_3. \tag{63}$$
Using $T \cdot N_2 = 0$ we have $a = 0$. Using $N_1 \cdot N_2 = 0$ we have $b_1 = -\kappa_2$. Using $\|N_2\| = 1$ we have $b_2 = 0$. Finally as
$$\begin{aligned} x^{(4)}(s) &= -(\kappa_1^2)' T - \kappa_1^2 T' + \kappa_1'' N_1 + \kappa_1' N_1' + (\kappa_1 \kappa_2)' N_2 + \kappa_1 \kappa_2 N_2' \\ &= -(\kappa_1^2)' T - \kappa_1^3 N_1 + \kappa_1'' N_1 + \kappa_1' (-\kappa_1 T + \kappa_2 N_2) \\ &\quad + (\kappa_1 \kappa_2)' N_2 + \kappa_1 \kappa_2 (-\kappa_2 N_1 + b_3 N_3), \end{aligned} \tag{64}$$
we conclude
$$\begin{aligned} \text{Volume of } \{x', x'', x''', x^{(4)}\} &= \text{Volume of } \{T, \kappa_1 N_1, \kappa_1 \kappa_2 N_2, \kappa_1 \kappa_2 b_3 N_3\} \\ &= \kappa_1^3 \kappa_2^2 b_3 \Longrightarrow b_3 = \kappa_3. \end{aligned} \tag{65}$$
Therefore
$$N_2' = -\kappa_2 N_1 + \kappa_3 N_3. \tag{66}$$
  - N_3'. Similarly we can show
$$N_3' = -\kappa_3 N_2 + \kappa_4 N_4. \tag{67}$$

Exercise 3. Derive the full Frenet-Serret equations.