Lectures 18–19: The Gauss-Bonnet Theorem

Disclaimer. As we have a textbook, this lecture note is for guidance and supplement only. It should not be relied on when preparing for exams.

In this lecture we introduce the Gauss-Bonnet theorem. The required section is §13.1. The optional sections are §13.2–§13.8.

I try my best to make the examples in this note different from examples in the textbook. Please read the textbook carefully and try your hands on the exercises. During this please don't hesitate to contact me if you have any questions.

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1. Gauss-Bonnet in the plane

1.1. The simplest cases

THEOREM 1. (GAUSS-BONNET FOR PLANE TRIANGLES) *Let* ABC *be a triangle in the flat plane. Then* $\angle A + \angle B + \angle C = \pi$.

THEOREM 2. (GAUSS-BONNET FOR PLANE CONVEX POLYGONS) *Let* $A_1 A_2 \ldots A_k$ *be a* k*-polygon in a plane. Further assume that it is convex. Then the sum of its exterior angles is* 2π.

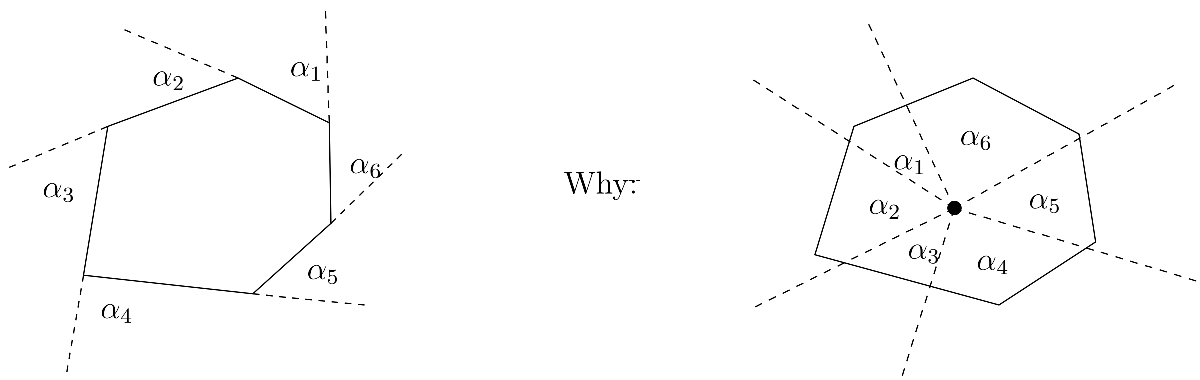

Figure 1. Sum of exterior angles of a convex polygon: $\alpha_1 + \cdots + \alpha_6 = 2\pi$

Remark 3. (SIGNED EXTERIOR ANGLES) The exterior angles of a plane polygon can be signed. Assume that we are traveling along the boundary counterclockwise. Then if the tangent vector is also turning counterclockwise, we say the exterior angle is positive, otherwise it's negative.

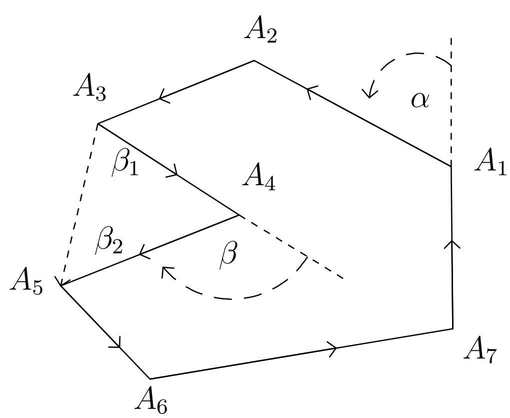

Figure 2. Positive and negative exterior angles: $\alpha > 0, \beta < 0$.

THEOREM 4. (GAUSS-BONNET FOR GENERAL PLANE POLYGONS) *The sum exterior angles of a plane polygon, convex or not, is* 2π.

"Proof". Such a general polygon is a convex polygon (its convex hull) with one or more convex polygon "taken away". The result follows from applying Gauss-Bonnet to each one of them.

Take the polygon in Figure 2 as an example. Applying Gauss-Bonnet to the small triangle $A_3 A_4 A_5$ we see that $\beta_1 + \beta_2 = \beta$ and therefore the sum of exterior angles of the non-convex polygon $A_1 \ldots A_7$ is the same as that of the convex polygon $A_1 A_2 A_3 A_5 A_6 A_7$.

1.2. Gauss-Bonnet for plane curvilinear polygons

- Angle = "concentrated curvature".

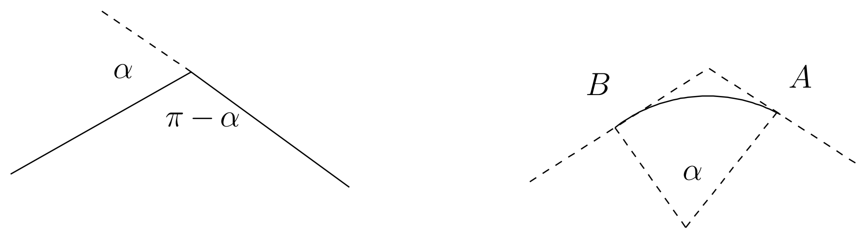

Figure 3. Angle as "concentrated curvature"

Exercise 1. We "smooth" the angle through an arc (part of a circle) that is tangent to the two "arms" at A, B respectively. Prove that

$$\int_A^B \kappa \, \mathrm{d}s = \alpha \tag{1}$$

where the integral is along the arc.

Exercise 2. What if we "smooth" the angle through a different family of arcs, say parabolas?

- Signed curvature for plane curves.

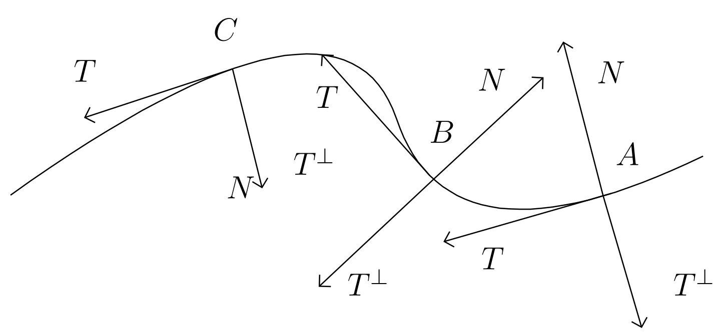

Figure 4. Signed curvature κ_s.

For the plane vector $T = (T_x, T_y)$, we can define $T^\perp := (-T_y, T_x)$ which is the counterclockwise rotation of T by $\pi/2$. Clearly $T^\perp \perp T$. On the other hand, we have $N = \kappa^{-1} \frac{\mathrm{d}T}{\mathrm{d}s} \perp T$. Thus either $N = T^\perp$ or $N = -T^\perp$. In the former case we define the signed curvature $\kappa_s = \kappa$ while in the latter case we define $\kappa_s = -\kappa$. For example in Figure 4 $\kappa_s = \kappa$ at C but $= -\kappa$ at A, B.

Exercise 3. Prove that $\kappa = |\kappa_s|$.

THEOREM 5. (GAUSS-BONNET FOR SIMPLE CLOSED PLANE CURVES) *A smooth, closed, non-self-intersecting planar curve* γ *satisfies*

$$\int_\gamma \kappa_s(s)\,\mathrm{d}s = 2\,\pi. \tag{2}$$

Proof.

i. We first prove the following. Let γ be parametrized by $x(s)\colon [a, b] \mapsto \mathbb{R}^2$ where s is the arc length parameter. Then

$$\int_a^b \kappa_s(s)\,\mathrm{d}s = \theta_0 + 2\,k\,\pi \tag{3}$$

for some $k \in \mathbb{Z}$.

To see this, notice that we can denote

$$T(s) = x'(s) = (\cos\theta(s), \sin\theta(s)) \tag{4}$$

where $\theta(s)$ is the angle between $x'(s)$ and $x'(a)$.[1] Taking derivative we have

$$x''(s) = \theta'(s)\,(-\sin\theta(s), \cos\theta(s)) = \theta'(s)\,T^{\perp}(s). \tag{5}$$

Therefore $\kappa_s(s) = \theta'(s)$. Consequently

$$\int_a^b \kappa_s(s)\,\mathrm{d}s = \theta(b). \tag{6}$$

Since $T(b) = (\cos\theta(b), \sin\theta(b)) = (\cos\theta_0, \sin\theta_0)$, (3) follows.

ii. Thus we see that for a closed simple curve there holds

$$\int_\gamma \kappa_s\,\mathrm{d}s = 2\,k\,\pi \tag{7}$$

for some $k \in \mathbb{Z}$. Now we prove that $k = 1$.

Wlog the parametrization is counterclociwise, that is the point $x(s)$ moves counterclockwise with respect to the interior of γ as s increases. Pick $s_0 = a < s_1 < \cdots < s_k < b = s_{k+1}$ such that

$$\int_{s_{i-1}}^{s_{i+1}} \kappa(s)\,\mathrm{d}s < \pi, \tag{8}$$

for all i. Then for any $s', s'' \in (s_{i-1}, s_{i+1})$, we have

$$\left| \int_{s'}^{s''} \kappa_s(s)\,\mathrm{d}s \right| \leqslant \int_{s'}^{s''} \kappa(s)\,\mathrm{d}s < \pi. \tag{9}$$

On the other hand, we have

$$\int_{s'}^{s''} \kappa_s(s)\,\mathrm{d}s = \delta\theta' + 2\,k\,\pi \tag{10}$$

1. Rigorously speaking, it is the angle between $x'(a)$ and the vector that is the parallel transported $x'(s)$.

where $\delta\theta' \in (-\pi, \pi]$ is the angle between $T(s'')$ and $T(s')$.

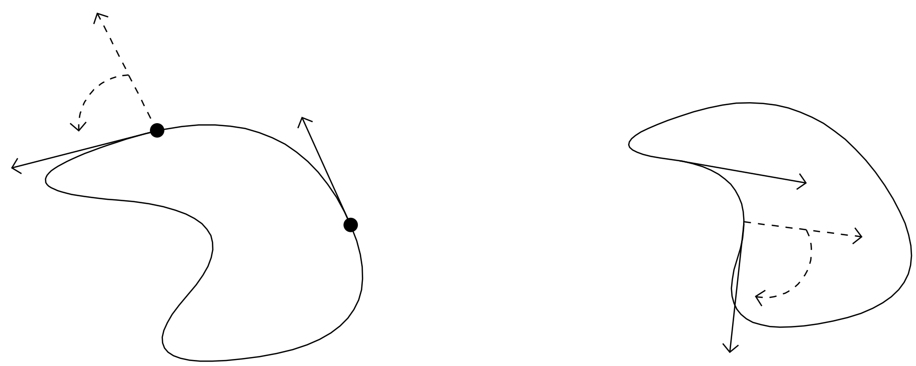

Figure 5. Left: $\delta\theta' > 0$; Right: $\delta\theta' < 0$.

Noticing that $|\delta\theta' + 2\,k\,\pi| > \pi$ for all $k \neq 0$, we conclude that

$$\int_{s'}^{s''} \kappa_s(s)\,\mathrm{d}s = \delta\theta', \tag{11}$$

when s', s'' are close enough.

Now let P be the polygon with vertices $x(s_0), x(s_1), \ldots, x(s_k), x(s_{k+1}) = x(s_0)$. By the mean value theorem[2] there are $s_0' \in [s_0, s_1], s_1' \in [s_1, s_2], \ldots, s_k' \in [s_k, s_{k+1}]$ such that $x'(s_i') \parallel \overrightarrow{x(s_i)x(s_{i+1})}$. Thanks to the arguments (8)–(11) we see that

$$\int_{s_i'}^{s_{i+1}'} \kappa_s(s)\,\mathrm{d}s = \text{the exterior angle at } x(s_i). \tag{12}$$

Now the conclusion (2) follows from Theorem 4, the the polygon Gauss-Bonnet theorem □

Remark 6. In the above proof, besides (8), we also require the partition $s_0, \ldots, s_k$ be such that the polygon P is simple, that is does not intersect itself.

Question 7. *Can this always be done through making $s_{i+1} - s_i$ small enough? If not, can we prove Gauss-Bonnet for polygons that intersect itself?*

Theorem 8. (Gauss-Bonnet for curvilinear polygons) *Let $\alpha_1, \ldots, \alpha_k$ be the exterior angles. Then*

$$\int_\gamma \kappa_s\,\mathrm{d}s + \sum_{i=1}^k \alpha_i = 2\,\pi. \tag{13}$$

Proof. Left as exercise. □

2. Keep in mind that the mean value theorem does not hold for space curves.

2. Surface Gauss-Bonnet

2.1. The plane curve case revisited: A mechanical point of view

The role of surface curvature can be understood through the following mechanical analogy.

- Curvature = centrifugal force.

  Consider a particle moving along a plane curve γ with unit speed. Then the position of this particle gives the arc length parametrization of γ: $x(s)$. Then the velocity and acceleration are

$$v(s) = x'(s), \qquad a(s) = x''(s). \tag{14}$$

  If we denote $n_s(s) := [x'(s)]^{\perp}$, there holds

$$a(s) = \kappa_s(s)\, n_s(s). \tag{15}$$

  Thus we see that $\kappa_s(s)$ is the "signed" magnitude of force. Consequently

$$2\pi = \int_\gamma \kappa_s(s)\, \mathrm{d}s = \text{“signed total” of work done.} \tag{16}$$

- Surface curvature = "Gravity" = "extra" centrifugal force.

  Now consider a particle moving along a surface curve. Then part of the the centrifugal force is provided by "gravity"–the force that keeps the particle on the surface. Thus we conjecture that

$$2\pi = \text{“signed total” of work by gravity} + \text{“signed total” of work by other forces.} \tag{17}$$

  Recall that on a surface, the trajectory of a particle moving under gravity only satisfies $\kappa_g = 0$ where κ_g is the geodesic curvature. On the other hand, the total work done by gravity should be related to the "total mass" enclosed by the curve. Thus we reach

$$2\pi = \int_\Omega \text{curvature}\, \mathrm{d}S + \int_\gamma \kappa_g(s)\, \mathrm{d}s \tag{18}$$

  where Ω is the part of the surface enclosed by γ.

  Exercise 4. Let S be a developable surface. Let γ be a curve on S. Let $\tilde{\gamma}$ be the curve on the plane that is the "flattened" S. Prove that for any $p \in S$ with $\tilde{p}$ the corresponding point on the plane, there holds $\kappa_g(p) = \kappa_s(\tilde{p})$.

Remark 9. There are other physical explanations for Gauss-Bonnet, for example see here. A more detailed version can be found in A "bicycle wheel" proof of the Gauss-Bonnet theorem, Mark Levi, Expo. Math. 12 (1994), 145–164.

2.2. Gauss-Bonnet on surfaces

Theorem 10. *Let S be a surface and $\gamma \subset S$ be a simple closed curve. Let Ω be the part of S that is enclosed by γ. There holds*

$$\int_\Omega K\, \mathrm{d}S + \int_\gamma \kappa_g\, \mathrm{d}s = 2\pi. \tag{19}$$

Exercise 5. Let S be the unit sphere. Let $\gamma \subset S$ be an arbitrary simple closed curve. Then γ divides S into two regions Ω_N, Ω_S. By Theorem 10 we have

$$\int_{\Omega_N} K \,\mathrm{d}S + \int_\gamma \kappa_g \,\mathrm{d}s = 2\pi = \int_{\Omega_S} K \,\mathrm{d}S + \int_\gamma \kappa_g \,\mathrm{d}s \Longrightarrow \int_{\Omega_N} K \,\mathrm{d}S = \int_{\Omega_S} K \,\mathrm{d}S \tag{20}$$

which means $\operatorname{area}(\Omega_N) = \operatorname{area}(\Omega_S)$. This is absurd. Explain.

Proof. We divide the proof of Theorem 10 into several steps.

i. Set-up. We parametrize γ as $x(s) = \sigma(u(s), v(s))$ where s is the arc length parameter. Let the range of s be from 0 to L. Denote by $W(s)$ a parallel tangent unit vector field along γ. Let $\theta(s)$ be the angle between $x'(s)$ and $W(s)$.

   Let $N_S(s)$ be the unit normal of S. Then we see that $W(s), N_S(s), W(s) \times N_S(s)$ form a right-handed orthonormal frame, and consequently

$$x'(s) = (\cos\theta(s))\, W(s) + (\sin\theta(s))\, W(s) \times N_S(s). \tag{21}$$

ii. The role of κ_g. Taking derivative of (21) we have

$$\begin{aligned} x''(s) &= \theta'(s)\,[(-\sin\theta(s))\, W(s) + (\cos\theta(s))W(s) \times N_S(s)] \\ &\quad + (\cos\theta(s))\, W'(s) + (\sin\theta(s))\, W'(s) \times N_S(s) \\ &\quad + (\sin\theta(s))\, W(s) \times N_S'(s). \end{aligned} \tag{22}$$

   As $W(s)$ is parallel along γ, we see that the black terms are tangent to T_pS, the grey term is zero, and the green terms are parallel to $N_S(s)$. Recalling the definition of the normal and geodesic curvatures, we see that

$$\kappa_g(s) = \theta'(s). \tag{23}$$

   Consequently, we have

$$\int_\gamma \kappa_g \,\mathrm{d}s = 2\pi - \Theta \tag{24}$$

   where Θ is the angle between $W(0)$ and $W(L)$.

iii. The role of K. Due to the presence of the surface curvature, we do not always have $W(0) = W(L)$, that is $\Theta = 0$, in (24).

   We take $\sigma(u, v)$ to be a geodesic surface patch, with first fundamental form $\mathrm{d}u^2 + \mathbb{G}(u,v)\,\mathrm{d}v^2$. Let $e_1 := \sigma_u, e_2 := \frac{\sigma_v}{\mathbb{G}^{1/2}}$. Then we have

$$W(s) = [\cos\theta(s)]\, e_1 + [\sin\theta(s)]\, e_2. \tag{25}$$

   This leads to

$$W'(s) = \theta'(s)\,[(-\sin\theta)\, e_1 + (\cos\theta)\, e_2] + (\cos\theta)\, e_1' + (\sin\theta)\, e_2'. \tag{26}$$

   As W is parallel along γ and is of unit length, we have $W' \perp [(-\sin\theta)\, e_1 + (\cos\theta)\, e_2]$. Thus

$$\begin{aligned} 0 &= W'(s) \cdot [(-\sin\theta)\, e_1 + (\cos\theta)\, e_2] \\ &= \theta'(s) + [(\cos\theta)^2\, e_1' \cdot e_2 - (\sin\theta)^2\, e_2' \cdot e_1] \\ &= \theta'(s) - e_2' \cdot e_1. \end{aligned} \tag{27}$$

Note that we have used $e_1' \cdot e_2 = -e_2' \cdot e_1$.

Now we have, setting γ' to be the closed plane curve $(u(s), v(s))$ and Ω' the region enclosed by γ', by Green's Theorem,

$$\begin{aligned} \int_0^L e_1 \cdot e_2' &= \int_0^L (e_1 \cdot e_{2,u})\, u' + (e_1 \cdot e_{2,v})\, v' \\ &= \int_{\gamma'} (e_1 \cdot e_{2,u})\, \mathrm{d}u + (e_1 \cdot e_{2,v})\, \mathrm{d}v \\ &= \int_{\Omega'} [e_{1,u} \cdot e_{2,v} - e_{1,v} \cdot e_{2,u}]\, \mathrm{d}u\, \mathrm{d}v. \end{aligned} \tag{28}$$

Substituting $e_1 := \sigma_u, e_2 := \frac{\sigma_v}{\mathbb{G}^{1/2}}$ into the above, we have the integrand to be

$$\sigma_{uu} \cdot \frac{\sigma_{vv}}{\mathbb{G}^{1/2}} - \frac{1}{2} \frac{(\sigma_{uu} \cdot \sigma_v)\, \mathbb{G}_v}{\mathbb{G}^{3/2}} - \frac{\sigma_{uv} \cdot \sigma_{uv}}{\mathbb{G}^{1/2}} + \frac{1}{2} \frac{(\sigma_{uv} \cdot \sigma_v)\, \mathbb{G}_u}{\mathbb{G}^{3/2}}. \tag{29}$$

As $\mathbb{E} = 1, \mathbb{F} = 0,$ we have

$$\Gamma_{11}^1 = \Gamma_{11}^2 = \Gamma_{12}^1 = 0, \Gamma_{12}^2 = \frac{\mathbb{G}_u}{2\,\mathbb{G}}, \Gamma_{22}^1 = -\frac{\mathbb{G}_u}{2}, \Gamma_{22}^2 = \frac{\mathbb{G}_v}{2\,\mathbb{G}}. \tag{30}$$

Consequently

$$\sigma_{uu} \cdot \sigma_{vv} = \mathbb{L}\,\mathbb{N}, \qquad \sigma_{uu} \cdot \sigma_v = 0, \tag{31}$$

$$\sigma_{uv} \cdot \sigma_{uv} = \frac{1}{4} \frac{\mathbb{G}_u^2}{\mathbb{G}^2} + \mathbb{M}^2, \qquad \sigma_{uv} \cdot \sigma_v = \frac{\mathbb{G}_u}{2\,\mathbb{G}}. \tag{32}$$

We see that

$$e_{1,u} \cdot e_{2,v} - e_{1,v} \cdot e_{2,u} = \frac{\mathbb{L}\,\mathbb{N} - \mathbb{M}^2}{\mathbb{G}^{1/2}} = K\,\sqrt{\mathbb{E}\,\mathbb{G} - \mathbb{F}^2}. \tag{33}$$

Therefore

$$\int_{\Omega'} [e_{1,u} \cdot e_{2,v} - e_{1,v} \cdot e_{2,u}]\, \mathrm{d}u\, \mathrm{d}v = \int_\Omega K\, \mathrm{d}S \tag{34}$$

and the proof ends. □

Remark 11. The proof of Theorem 10 here is not fully rigorous (can you spot the gaps?). Yet it is intuitive and consistent with our proof in the plane case.

Exercise 6. Read through the proof in §13.1 of the textbook and understand every detail.

Remark 12. By (19) it is easy to see that if γ is a closed geodesic, then necessarily $\int_\Omega K\, \mathrm{d}S = 2\,\pi$. Consequently there is no closed geodesic on a surface with $K \leqslant 0$ everywhere.

Exercise 7. Let S be a cylinder. Then clearly there are closed geodesics. Can you explain this?

THEOREM 13. (CURVILINEAR POLYGONS ON A SURFACE) *For a curvilinear polygon on a surface S, we have*

$$2\,\pi = \int_\gamma \kappa_g\, \mathrm{d}s + \sum \alpha_i + \int_\Omega K\, \mathrm{d}S \tag{35}$$

where α_i are the exterior angles at the vertices.[3]

3. Gauss-Bonnet on compact surfaces

3.1. Integration on compact surfaces

- Recall that we can integrate on a surface patch through

$$\int_S f\,\mathrm{d}S = \int_U f(\sigma(u,v))\,\|\sigma_u\times\sigma_v\|\,\mathrm{d}u\,\mathrm{d}v. \tag{36}$$

What if the surface cannot be covered by one single surface patch? In particular, how do we integrate on a compact surface S?

Exercise 8. Show that a compact surface cannot be covered by one single surface patch.

- The idea is "partition of unity". Assume that S is covered by N surface patches $\sigma_1, \ldots, \sigma_N$, where $\sigma_i: U_i \mapsto S$ with $\Omega_i = \sigma(U_i)$. Note that each Ω_i is open and $\cup_{i=1}^N \Omega_i = S$.

For every Ω_i, let $\tilde{\Omega}_i := \Omega_i - \cup_{j\neq i}\Omega_j$. Then $\tilde{\Omega}_i$ is closed. Let $\sigma\big(\tilde{U}_i\big) = \tilde{\Omega}_i$. We see that $\varepsilon_i := \operatorname{dist}\big(\tilde{U}_i, \partial U_i\big)/3 > 0$.[4] We define $W_i := \big\{x \in U_i |\ \operatorname{dist}\big(x, \tilde{U}_i\big) \leqslant \varepsilon_i\big\}$ and $\tilde{W}_i := \big\{x \in U_i |\ \operatorname{dist}\big(x, \tilde{U}_i\big) \leqslant 2\,\varepsilon_i\big\}$.

Next take a smooth even function $\rho \geqslant 0$ such that

$$2\,\pi\int_0^\infty \rho(t)\,t\,\mathrm{d}t = 1, \qquad \rho(t) = \begin{cases} 1 & |t| < 1/4 \\ 0 & |t| > 3/4 \end{cases}. \tag{37}$$

We see that the function $\phi(u,v) := \rho\big(\sqrt{u^2+v^2}\big)$ satisfies

$$\int_{\mathbb{R}^2} \phi(u,v)\,\mathrm{d}u\,\mathrm{d}v = 2\,\pi\int_0^\infty \rho(r)\,r\,\mathrm{d}r = 1 \tag{38}$$

and $\phi(u,v) = \begin{cases} 1 & \sqrt{u^2+v^2} < 1/4 \\ 0 & \sqrt{u^2+v^2} > 3/4 \end{cases}$. Now define

$$\phi_i(u,v) := \frac{1}{\varepsilon_i^2}\,\phi\left(\frac{u}{\varepsilon_i}, \frac{v}{\varepsilon_i}\right). \tag{39}$$

Let the function $\chi_i(u,v) := \begin{cases} 1 & (u,v) \in W_i \\ 0 & (u,v) \notin W_i \end{cases}$. Define

$$\Phi_i(u,v) := \int_{\mathbb{R}^2} \phi_i(u-u', v-v')\,\chi_i(u',v')\,\mathrm{d}u'\,\mathrm{d}v'. \tag{40}$$

Then $\Phi_i(u,v)$ is smooth and satisfy

$$\Phi_i(u,v) = \begin{cases} 1 & (u,v) \in \tilde{U}_i \\ >0 & (u,v) \in \tilde{W}_i \\ =0 & (u,v) \text{ outside } \tilde{W}_i \end{cases}. \tag{41}$$

3. Note that our α_i here are different from those in §13.2 of the textbook.

4. If $U_i = \mathbb{R}^2$ just set $\varepsilon_i = 1$.

Finally define $\Psi_i = \frac{\Phi_i \circ \sigma_i^{-1}}{\sum_{j=1}^N \Phi_j \circ \sigma_j^{-1}}$. We see that

$$\sum_{i=1}^N \Psi_i = 1 \text{ all over } S. \tag{42}$$

Such $\{\Psi_i\}$ is called a "partition of unity" of S.

- With such "partition of unity" available, we can simply define

$$\int_S f \,\mathrm{d}S := \sum_{i=1}^N \int_{U_i} F_i(\sigma_i(u, v)) \, \|\sigma_{i,u} \times \sigma_{i,v}\| \,\mathrm{d}u \,\mathrm{d}v \tag{43}$$

where

$$F_i = f\, \Psi_i. \tag{44}$$

3.2. Euler number

Definition 14. *Let P be a polygon. Define the* **Euler number** χ *as*

$$\chi = V - E + F \tag{45}$$

where V is the number of vertices, E the number of edges, and F the number of faces.

Remark 15. It turns out that χ is a "topological invariant". It is easy to convince ourselves that deforming a polygon would not change χ. Thus χ depends only on the "shape" of the polygon. A few examples.

- If we can "blow up" the polygon into a sphere, then $\chi = 2$. In other words, $\chi(\text{sphere}) = 2$.

  To see this, we do the following operations.

  i. Take away one face and "flatten" the the "polytope with a hole". Thus $F \mapsto F - 1$ and E, V remain the same.

  ii. Let e be any edge that is not on the boundary. There are two situations.

     a) Both ends of e have more than two edges connected to the vertices. In this case we take e away. then the two adjacent faces are merged together. Thus after this operation we have $E \mapsto E - 1$ and $F \mapsto F - 1$, while V remains the same.

     b) One or both ends of e is connected to one other edge. Then we merge this edge with e. When there is only one such end, the result is $E \mapsto E - 1, V \mapsto V - 1$, when there are two such ends, we have $E \mapsto E - 2$, $V \mapsto V - 2$.

     Note that in either case, $V - E + F$ stays unchanged.

  iii. Keep doing step ii until there is no interior edge anymore. Then we would have a polygon, for which $V = E, F = 1$. Thus the original χ should be $1 + 1 = 2$.

- $\chi(\text{torus}) = 0$.

The key difference here is that we still cannot flatten the "polytope with a hole" after "taking one face away". Intuitively, if we "cut" the torus and "straighten" it into a cylinder, then $V-E$ stays the same while $F\mapsto F+2$. But a cylinder (of finite height) is topologically equivalent to the sphere so

$$V+F+2-E=2\Longrightarrow\chi=V-E+F=0 \tag{46}$$

for the torus.

- $\chi(\text{cup with handle})=-2.$

  This is equivalent to connecting two torus together.

3.3. Gauss-Bonnet on compact surfaces

THEOREM 16. [5]*Let S be a compact surface. Then*

$$\int_S K\,\mathrm{d}S=2\,\pi\,\chi. \tag{47}$$

Remark 17. If S is an apple, then the total Gaussian curvature is $2\,\pi$. Now take a pen to poke it. During the process the total Gaussian curvature stays $2\,\pi$. But the moment you poke it through, it becomes 0.

Proof. We sketch the idea. Intuitively, we can divide S into finitely many triangles $T_1,\ldots,$ T_F and thus S becomes a "curvilinear polygon" with F faces. We note that as each face has three edges and each edge is shared by two faces, there holds $E=\frac{3\,F}{2}$. On each triangle we apply Theorem 13:

$$\int_{T_i}K\,\mathrm{d}S+\int_{e_{i1}\cup e_{i2}\cup e_{i3}}\kappa_g\,\mathrm{d}s+\alpha_{i1}+\alpha_{i2}+\alpha_{i3}=2\,\pi \tag{48}$$

where e_{i1}, e_{i2}, e_{i3} are the three edges and α_{i1}, α_{i2}, α_{i3} are the three exterior angles. Now summing over $i=1,2,\ldots,F$ we see that[6]

$$\sum_i\int_{T_i}K\,\mathrm{d}S=\int_S K\,\mathrm{d}S,\qquad \sum_i\int_{e_{i1}\cup e_{i2}\cup e_{i3}}\kappa_g\,\mathrm{d}s=0. \tag{49}$$

Now we sum up the exterior angles through a different way of counting. Let $A_1,\ldots,A_V$ be the vertices. Denote by E_i the number of edges connected to each A_i. Then we see that,

$$\text{sum of exterior angles at } A_i=E_i\,\pi-\sum\text{interior angles at }A_i=(E_i-2)\,\pi. \tag{50}$$

Therefore (note that each edge connects two vertices)

$$\begin{aligned}\sum\text{all exterior angles}-2\,F\pi &= \sum_{i=1}^V(E_i-2)\,\pi-2\,F\pi\\ &= \left(\sum_{i=1}^V E_i\right)\pi-2\,V\pi-2\,F\pi\\ &= 2\,E\pi-2\,V\pi-2\,F\pi\\ &= 2\,\pi\,(E-V-F)=-2\,\pi\,\chi.\end{aligned} \tag{51}$$

5. Theorem 13.4.5 of the textbook.

6. Intuitively, we can simply take the edges to be geodesics, then $\kappa_g=0$ and the edge terms vanish.

The conclusion then follows. □

Remark 18. We see that the Gaussian curvature is invariant under local isometries, but the integral of the Gaussian curvature over a compact surface is even more invariant–it only depends on the "shape" of the surface.