Lectures 14–15: Geodesics

Disclaimer. As we have a textbook, this lecture note is for guidance and supplement only. It should not be relied on when preparing for exams.

In this lecture we study the shortest path connecting two points in a surface.

The required textbook sections are §7.4, §9.1–9.4. The optional sections are §9.5

I try my best to make the examples in this note different from examples in the textbook. Please read the textbook carefully and try your hands on the exercises. During this please don't hesitate to contact me if you have any questions.

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1. Parallel transport

1.1. Definition

- In classical geometry, one of the most important ideas is parallelism.
- For example, $\mathbb{R}^2$ or $\mathbb{R}^3$, the shortest paths are straight lines, which are characterized by the fact that the tangent direction never changes. But what does this mean? In classical geometry we show that the two tangent directions at different points on the path are parallel to each other.
- However on a curved surface this becomes problematic. For example consider the following situation:

  Consider the unit sphere. Let B be the north pole and A, B be two points on the equator. Then clearly v_B should be the tangent vector at B that is "parallel" to v_A at A. Similarly $v_C \parallel v_B$, $\tilde{v}_A \parallel v_C$. However it is clear that $v_A \nparallel \tilde{v}_A$.

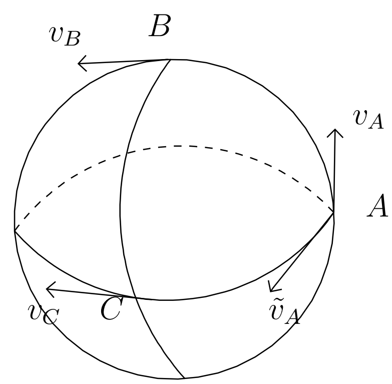

Figure 1. $v_A \parallel v_B, v_B \parallel v_C, v_C \parallel \tilde{v}_A$.

- Therefore it does not make much sense on a curved surface to say "two vectors at two different points are parallel to each other".
- On the other hand, if we adopt a more "dynamical" interpretation of parallelism, it becomes possible to adapt this notion to curved surfaces.
- Instead of comparing vectors at two different points, we consider the following situation.

  Let γ be a curve on a surface S. Let w be a tangent vector field along γ, that is $w: S \mapsto \mathbb{R}^3$ such that $w(p) \in T_p S$ for every $p \in \gamma$.

Remark 1. As soon as γ is parametrized by some $x(t)$, we can form the composite function $w(x(t))$. When no confusion should arise, we abuse notation a bit and simply write $w(t)$.

Remark 2. It is easy to see how "a tangent vector field on S" should be defined.

Exercise 1. Give a reasonable definition to a "tangent vector field on S".

We would like to give definition to "w does not change direction along γ".

- One reasonable definition is the following.
  - Covariant derivative.

    In the above setting, the covariant derivative of w along γ is given by the tangential component of w':

$$\nabla_\gamma w = w' - (w' \cdot N_S)\, N_S \tag{1}$$

    where N_S is the unit normal of the surface.
  - Then we say v to be parallel along γ if $\nabla_\gamma w = 0$ at every point of γ.

Remark 3. Clearly,

$$\nabla_\gamma w = 0 \Longleftrightarrow w' \perp T_{x(t)} S. \tag{2}$$

1.2. Examples

Example 4. Let S be the x-y plane. Let $x(t) = (u(t),\, v(t),\, 0)$. Then we have $w(t) := x'(t) = (u'(t), v'(t), 0)$ and

$$\nabla_\gamma w(t) = x''(t) - [x''(t) \cdot N_S(t)] N_S(t) = (u''(t), v''(t), 0) \tag{3}$$

as $N_S(t) = (0, 0, 1)$ for all t. Consequently $x'(t)$ is parallel along γ if and only if $u''(t) = v''(t) = 0$, that is $u = a_1\, t + a_0, v = b_1\, t + b_0$.

Thus a plane curve is "straight" when it is a straight line.

Example 5. Let S be the cylinder $\sigma(u, v) = (\cos u, \sin u, v)$. Let γ be a curve on S. Thus γ can be parametrized as $x(t) = (\cos u(t), \sin u(t), v(t))$. We have

$$w(t) := x'(t) = ((-\sin u(t))\, u'(t), (\cos u(t))\, u'(t), v'(t)) \tag{4}$$

and

$$x''(t) = ((-\cos u)\, u'^2 - (\sin u)\, u'', (-\sin u)\, u'^2 + (\cos u)\, u'', v''). \tag{5}$$

On the other hand, we have

$$\sigma_u = (-\sin u, \cos u, 0), \qquad \sigma_v = (0, 0, 1), \tag{6}$$

therefore

$$N_S = \frac{\sigma_u \times \sigma_v}{\|\sigma_u \times \sigma_v\|} = (\cos u, \sin u, 0). \tag{7}$$

Thus we can calculate

$$\nabla_\gamma w(t) = (-(\sin u)\, u'', (\cos u)\, u'', v''). \tag{8}$$

Therefore $\nabla_\gamma w(t) = 0 \Longleftrightarrow$

$$-(\sin u)\, u'' = 0, \qquad (\cos u)\, u'' = 0, \qquad v'' = 0. \tag{9}$$

This is equivalent to $u'' = 0, v'' = 0$.

Thus a cylindrical curve is "straight" when it is of the form $(\cos u(t), \sin u(t), v(t))$ where $(u(t), v(t))$ is a straight line in the plane.

Example 6. Let S be the unit sphere given by $\sigma(u,v)=(\cos u\cos v,\cos u\sin v,\sin u)$. We consider the curve γ to be a circle parametrized as $x(t)=(\cos u_0\cos t,\cos u_0\sin t,\sin u_0)$. Let $w(t)=x'(t)$ be the tangent vector. We have

$$w'(t)=(-\cos u_0\cos t,-\cos u_0\sin t,0). \tag{10}$$

On the other hand we have

$$N_S(u,v)=(\cos u\cos v,\cos u\sin v,\sin u). \tag{11}$$

Therefore

$$\nabla_\gamma w(t)=\frac{\sin 2\,u_0}{2}\,(-\sin u_0\cos t,-\sin u_0\sin t,\cos u_0). \tag{12}$$

We see that it is zero only if $u_0=0$, that is γ is part of a big circle.

Exercise 2. What about $u_0=\pi/2$?

Example 7. Of course we should not restrict ourselves to the tangent of the curve. We take the setting of Example 6 and let $w(t):=(-\sin u_0\cos t,-\sin u_0\sin t,\cos u_0)$ be the unit tangent vector at $x(t)$ “pointing north”.

We have

$$w'(t)=(\sin u_0\sin t,-\sin u_0\cos t,0). \tag{13}$$

Again

$$N_S(u,v)=(\cos u\cos v,\cos u\sin v,\sin u). \tag{14}$$

Therefore

$$\nabla_\gamma w(t)=\sin u_0\,(\sin t,-\cos t,0). \tag{15}$$

Again $w(t)$ is parallel along γ if and only if $u_0=0$, that is γ is part of a big circle.

Exercise 3. Study $\nabla_\gamma w(t)$ for $w(t)=x'(t)$ for an arbitrary spherical curve.

1.3. Properties

Lemma 8. *Let γ be a curve on S. Let w be a tangent vector field along γ. Then the condition $\nabla_\gamma w=0$ is independent of the parametrization of γ.*

Proof. Let $x(t),\tilde{x}(\tilde{t})$ be two different parametrizations of γ. Then there is a function $\tilde{T}(t)$ such that $\tilde{x}(\tilde{t})=x(\tilde{T}(t))$. Since w is a vector field along γ, we have

$$\tilde{w}(\tilde{t})=w(\tilde{T}(t)). \tag{16}$$

Consequently

$$\nabla_\gamma\tilde{w}=w'\,\tilde{T}'-\left(w'\,\tilde{T}'\cdot N_S\right)N_S=\tilde{T}'\nabla_\gamma w. \tag{17}$$

Therefore $\nabla_\gamma w=0\Longleftrightarrow\nabla_\gamma\tilde{w}=0$. $\square$

Exercise 4. Is $\nabla_\gamma w$ independent of the parametrization of γ?

Remark 9. Lemma 8 justifies the notation ∇_γ where parametrization is not involved. The situation can be further simplified by the following lemma, which says that covariant derivative is simply "directional derivative" on surfaces.

LEMMA 10. *Let $\gamma, \tilde{\gamma}$ be two curves on S that are tangent at $p \in S$. Let w be a tangent vector field of S, that is w: $S \mapsto \mathbb{R}^3$ with $w(p) \in T_p S$. Let γ, $\tilde{\gamma}$ be parametrized by $x(t)$, $\tilde{x}(\tilde{t})$ with $p = x(t_0) = \tilde{x}(\tilde{t}_0)$ and furthermore $x'(t_0) = \tilde{x}'(\tilde{t}_0)$. Then $\nabla_\gamma w = \nabla_{\tilde{\gamma}} w$ at p.*

Proof. We have

$$\frac{\mathrm{d}w}{\mathrm{d}t}(t_0) = D_p w(x'(t_0)) = D_p w(\tilde{x}'(\tilde{t}_0)) = \frac{\mathrm{d}w}{\mathrm{d}\tilde{t}}(\tilde{t}_0). \tag{18}$$

Consequently

$$\nabla_\gamma w = \frac{\mathrm{d}w}{\mathrm{d}t}(t_0) - \left[\frac{\mathrm{d}w}{\mathrm{d}t}(t_0) \cdot N_S\right] N_S = \frac{\mathrm{d}w}{\mathrm{d}\tilde{t}}(\tilde{t}_0) - \left[\frac{\mathrm{d}w}{\mathrm{d}\tilde{t}}(\tilde{t}_0) \cdot N_S\right] N_S = \nabla_{\tilde{\gamma}} w, \tag{19}$$

exactly what we need to prove. □

LEMMA 11. *Let γ be a curve on S parametrized as $x(t)$. Then the following are equivalent.*

i. Along γ there holds $\kappa(t) = |\kappa_n|$;

ii. Along γ there holds $\kappa_g = 0$;

iii. $T(t)$, the unit tangent vector to γ, is parallel along γ.

Remark 12. Thus the three seemingly different ways to characterize "as straight as possible" curves on a curved surface,

1. $\kappa(t) = |\kappa_n(x(t))|$,
2. $\kappa_g(t) = 0$;
3. $\nabla_\gamma T(t) = 0$,

are all equivalent.

Proof. Thanks to Lemma 8, we can take $x(s)$ to be the arc length parametrization of γ. Then recall that by definition of κ_n, κ_g we have

$$x''(s) = \kappa_n N_S + \kappa_g (T \times N_S). \tag{20}$$

Consequently

$$\nabla_\gamma T = x''(s) - \kappa_n N_S, \tag{21}$$

and the conclusion follows. □

Exercise 5. There is a minor gap in the above argument. Can you fix it?

1.4. Calculation of the covariant derivative and Christoffel symbols

- How to calculate covariant derivative on an abstract surface, with only the two fundamental forms given?

- Set up. Let S be a surface parametrized by the patch $\sigma(u,v)$. Let $\gamma: x(t) = \sigma(u(t), v(t))$ and $w = w(u,v)$ be a tangent vector field along γ. Therefore there are $\alpha(t), \beta(t)$ such that $w = \alpha\,\sigma_u + \beta\,\sigma_v$.

- Now we calculate

$$\begin{aligned} w' &= \alpha'\,\sigma_u + \beta'\,\sigma_v + \alpha\,[\sigma_{uu}\,u' + \sigma_{uv}\,v'] + \beta\,[\sigma_{vu}\,u' + \sigma_{vv}\,v'] \\ &= \alpha'\,\sigma_u + \beta'\,\sigma_v + (\alpha\,u')\,\sigma_{uu} + (\alpha\,v' + \beta\,u')\,\sigma_{uv} + (\beta\,v')\,\sigma_{vv}. \end{aligned} \tag{22}$$

Therefore

$$\begin{aligned} \nabla_\gamma w &= w' - (w' \cdot N_S)\,N_S \\ &= \alpha'\,\sigma_u + \beta'\,\sigma_v \\ &\quad + (\alpha\,u')\,(\sigma_{uu} - (\sigma_{uu}\cdot N_S)\,N_S) \\ &\quad + (\alpha\,v' + \beta\,u')\,(\sigma_{uv} - (\sigma_{uv}\cdot N_S)\,N_S) \\ &\quad + (\beta\,v')\,(\sigma_{vv} - (\sigma_{vv}\cdot N_S)\,N_S). \end{aligned} \tag{23}$$

- To understand this formula we introduce Christoffel symbols Γ_{ij}^k and the related Gauss equations.

Proposition 13. (Gauss Equations) *Let $\sigma(u,v)$ be a surface patch with first and second fundamental forms*

$$\mathbb{E}\,\mathrm{d}u^2 + 2\,\mathbb{F}\,\mathrm{d}u\,\mathrm{d}v + \mathbb{G}\,\mathrm{d}v^2 \text{ and } \mathbb{L}\,\mathrm{d}u^2 + 2\,\mathbb{M}\,\mathrm{d}u\,\mathrm{d}v + \mathbb{N}\,\mathrm{d}v^2. \tag{24}$$

Then

$$\sigma_{uu} = \Gamma_{11}^1\,\sigma_u + \Gamma_{11}^2\,\sigma_v + \mathbb{L}\,N, \tag{25}$$

$$\sigma_{uv} = \Gamma_{12}^1\,\sigma_u + \Gamma_{12}^2\,\sigma_v + \mathbb{M}\,N, \tag{26}$$

$$\sigma_{vv} = \Gamma_{22}^1\,\sigma_u + \Gamma_{22}^2\,\sigma_v + \mathbb{N}\,N, \tag{27}$$

where

$$\begin{array}{ll} \Gamma_{11}^1 = \dfrac{\mathbb{G}\,\mathbb{E}_u - 2\,\mathbb{F}\,\mathbb{F}_u + \mathbb{F}\,\mathbb{E}_v}{2\,(\mathbb{E}\,\mathbb{G} - \mathbb{F}^2)}, & \Gamma_{11}^2 = \dfrac{2\,\mathbb{E}\,\mathbb{F}_u - \mathbb{E}\,\mathbb{E}_v + \mathbb{F}\,\mathbb{E}_u}{2\,(\mathbb{E}\,\mathbb{G} - \mathbb{F}^2)}, \\ \Gamma_{12}^1 = \dfrac{\mathbb{G}\,\mathbb{E}_v - \mathbb{F}\,\mathbb{G}_u}{2\,(\mathbb{E}\,\mathbb{G} - \mathbb{F}^2)}, & \Gamma_{12}^2 = \dfrac{\mathbb{E}\,\mathbb{G}_u - \mathbb{F}\,\mathbb{E}_v}{2\,(\mathbb{E}\,\mathbb{G} - \mathbb{F}^2)}, \\ \Gamma_{22}^1 = \dfrac{2\,\mathbb{G}\,\mathbb{F}_v - \mathbb{G}\,\mathbb{G}_u - \mathbb{F}\,\mathbb{G}_v}{2\,(\mathbb{E}\,\mathbb{G} - \mathbb{F}^2)}, & \Gamma_{22}^2 = \dfrac{\mathbb{E}\,\mathbb{G}_v - 2\,\mathbb{F}\,\mathbb{F}_v + \mathbb{F}\,\mathbb{G}_u}{2\,(\mathbb{E}\,\mathbb{G} - \mathbb{F}^2)}. \end{array} \tag{28}$$

The six Γ coefficients in these formulas are called Christoffel symbols.

Remark 14. The formulas (28) look very complicated. However we will see in the proof below that it is not hard to derive them "on the fly".

Proof. First note that as $\{\sigma_u, \sigma_v, N\}$ form a basis of $\mathbb{R}^3$ at p, there must exist nine numbers such that (25–27) hold. Take inner product of (25–27) with N we see that the coefficients for N must be $\mathbb{L}, \mathbb{M}, \mathbb{N}$.

Now consider (25). Taking inner product with σ_u and σ_v we have

$$\begin{aligned} \mathbb{E}\,\Gamma_{11}^1 + \mathbb{F}\,\Gamma_{11}^2 &= \sigma_{uu} \cdot \sigma_u = \left(\frac{\mathbb{E}}{2}\right)_u, & (29)\\ \mathbb{F}\,\Gamma_{11}^1 + \mathbb{G}\,\Gamma_{11}^2 &= \sigma_{uv} \cdot \sigma_v = \left(\frac{\mathbb{G}}{2}\right)_u. & (30)\end{aligned}$$

The first line of formulas in (28) immediately follows. The proofs for the other four formulas are similar and left as exercise. □

- With the help of Christoffel symbols, we can characterize conditions for a tangent vector field $w(t) := \alpha(t)\,\sigma_u + \beta(t)\,\sigma_v$ to be parallel along a curve $x(t) = \sigma(u(t), v(t))$.

THEOREM 15. [1]*$w(t)$ is parallel along $x(t)$ if and only if the following equations are satisfied:*

$$\begin{aligned} \alpha' + (\Gamma_{11}^1\,u' + \Gamma_{12}^1\,v')\,\alpha + (\Gamma_{12}^1\,u' + \Gamma_{22}^1\,v')\,\beta &= 0,\\ \beta' + (\Gamma_{11}^2\,u' + \Gamma_{12}^2\,v')\,\alpha + (\Gamma_{12}^2\,u' + \Gamma_{22}^2\,v')\,\beta &= 0. \end{aligned} \qquad (31)$$

Proof. This follows easily from (25–27). □

Remark 16. Note that the above equations are easier to remember in matrix form:

$$\alpha' + \left[\begin{pmatrix} \Gamma_{11}^1 & \Gamma_{12}^1 \\ \Gamma_{12}^1 & \Gamma_{22}^1 \end{pmatrix}\begin{pmatrix} u' \\ v' \end{pmatrix}\right] \cdot \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = 0, \qquad (32)$$

and

$$\beta' + \left[\begin{pmatrix} \Gamma_{11}^2 & \Gamma_{12}^2 \\ \Gamma_{12}^2 & \Gamma_{22}^2 \end{pmatrix}\begin{pmatrix} u' \\ v' \end{pmatrix}\right] \cdot \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = 0. \qquad (33)$$

Remark 17. Also keep in mind that when we "upgrade" to Riemannian geometry, a "surface" will not be given as a "surface patch" with explicit formulas, but as a collection of quantities defined at every $p \in S$: $\mathbb{E}, \mathbb{F}, \mathbb{G}, \mathbb{L}, \mathbb{M}, \mathbb{N}$ and Γ_{ij}^k.

Observe that in (31) only the first fundamental form and the tangent direction $x'(t)$ are involved.

- Examples and remarks.

Example 18. Let S be the x-y plane, parametrized by $\sigma(u, v) = (u, v, 0)$. Then we easily have $\Gamma_{ij}^k = 0$ for all i, j, k. (31) now becomes

$$\sigma' = \beta' = 0. \qquad (34)$$

Just as we expected.

Remark 19. The Christoffel symbol Γ_{ij}^k is roughly the k-th component of the change of the i-th coordinate vector along the j-th direction.

1. Proposition 7.4.5 of the textbook.

Note that if $\Gamma_{ij}^k = 0$ for all i, j, k, then the coordinate vectors σ_u, σ_v are parallel along $u = \text{const}$ and $v = \text{const}$.

Example 20. Let S be the cylinder $(\cos u, \sin u, v)$. Then we have

$$\sigma_u = (-\sin u, \cos u, 0), \qquad \sigma_v = (0, 0, 1) \tag{35}$$

and

$$\mathbb{E} = 1, \qquad \mathbb{F} = 0, \qquad \mathbb{G} = 1. \tag{36}$$

Thus $\Gamma_{ij}^k = 0$ for all i, j, k. (31) again gives

$$\sigma' = \beta' = 0. \tag{37}$$

Example 21. Let S be the unit sphere $(\cos u \cos v, \cos u \sin v, \sin u)$. We have

$$\mathbb{E} = 1, \qquad \mathbb{F} = 0, \qquad \mathbb{G} = \cos^2 u. \tag{38}$$

This leads to

$$\begin{aligned} \Gamma_{11}^1 &= \frac{\mathbb{G}\,\mathbb{E}_u - 2\,\mathbb{F}\,\mathbb{F}_u + \mathbb{F}\,\mathbb{E}_v}{2\,(\mathbb{E}\,\mathbb{G} - \mathbb{F}^2)} = 0, & \Gamma_{11}^2 &= \frac{2\,\mathbb{E}\,\mathbb{F}_u - \mathbb{E}\,\mathbb{E}_v + \mathbb{F}\,\mathbb{E}_u}{2\,(\mathbb{E}\,\mathbb{G} - \mathbb{F}^2)} = 0, \\ \Gamma_{12}^1 &= \frac{\mathbb{G}\,\mathbb{E}_v - \mathbb{F}\,\mathbb{G}_u}{2\,(\mathbb{E}\,\mathbb{G} - \mathbb{F}^2)} = 0, & \Gamma_{12}^2 &= \frac{\mathbb{E}\,\mathbb{G}_u - \mathbb{F}\,\mathbb{E}_v}{2\,(\mathbb{E}\,\mathbb{G} - \mathbb{F}^2)} = -\tan u, \\ \Gamma_{22}^1 &= \frac{2\,\mathbb{G}\,\mathbb{F}_v - \mathbb{G}\,\mathbb{G}_u - \mathbb{F}\,\mathbb{G}_v}{2\,(\mathbb{E}\,\mathbb{G} - \mathbb{F}^2)} = \sin u \cos u, & \Gamma_{22}^2 &= \frac{\mathbb{E}\,\mathbb{G}_v - 2\,\mathbb{F}\,\mathbb{F}_v + \mathbb{F}\,\mathbb{G}_u}{2\,(\mathbb{E}\,\mathbb{G} - \mathbb{F}^2)} = 0. \end{aligned} \tag{39}$$

(31) now becomes

$$\alpha' + (\sin u \cos u)\, v'\, \beta = 0, \qquad \beta' - (\tan u)\, v'\, \alpha = 0. \tag{40}$$

Thus w is parallel along γ if and only if (40) holds.

1.5. Parallel transport map

Definition 22. *Let $p, q \in S$ and let γ be a curve on S parametrized by $x(t)$ connecting p, q with $p = x(t_0), q = x(t_1)$. Let $w_0 \in T_p S$. Then there is a unique vector field $w(t)$ parallel along γ with $w(t_0) = w_0$. The map $\Pi_\gamma^{pq}: T_p S \mapsto T_q S$ taking w_0 to $w(t_1)$ is called* parallel transport *from p to q along γ.*

Proposition 23. *Π_γ^{pq} is an isometry.*

Proof. We have

$$(w \cdot \tilde{w})' = w' \cdot \tilde{w} + w \cdot \tilde{w}' = 0. \tag{41}$$

as $w', \tilde{w}' \parallel N$. Therefore for $w_0, \tilde{w}_0 \in T_p S$,

$$\Pi_\gamma^{pq}(w_0) \cdot \Pi_\gamma^{pq}(\tilde{w}_0) - w_0 \cdot \tilde{w}_0 = \int_{t_0}^{t_1} (w \cdot \tilde{w})'\, \mathrm{d}t = 0, \tag{42}$$

and the conclusion follows. □

Example 24. Let S be the unit sphere $(\cos u \cos v, \cos u \sin v, \sin u)$. We have seen that a vector field $\alpha(t)\,\sigma_u + \beta(t)\,\sigma_v$ is parallel along γ is equivalent to

$$\alpha' + (\sin u \cos u)\, v'\,\beta = 0, \qquad \beta' - (\tan u)\, v'\,\alpha = 0. \tag{43}$$

Now notice that unless $\sin u = 0$, that is γ is the big circle, the solution does not satisfy $\alpha' = \beta' = 0$.

2. Geodesics

2.1. Definition and basic properties

We have seen that there are three equivalent ways to characterize a curve γ on a surface S being "as straight as possible" curves on a curved surface,

1. Curvature of the curve at $p \in \gamma$ equals $|\kappa_n(p)|$ where $\kappa_n(p)$ is the normal curvature of S at p in the tangent direction of γ.
2. The geodesic curvature of the curve is zero, that is $\kappa_g(t) = 0$;
3. The covariant derivative of the unit tangent vector of the curve is zero along the curve, that is $\nabla_\gamma T = 0$.

Now we give a name to these "as straight as possible" curves.

DEFINITION 25. (GEODESICS)

- *A curve γ on the surface S is called a geodesic if $\nabla_\gamma T = 0$ where T is the unit tangent vector of γ.*
- *A parametrized curve $x(t)$ on the surface S is called a geodesic if $\nabla_\gamma x'(t) = 0$.*

PROPOSITION 26. (SOME BASIC PROPERTIES)

i. [2]*Let $x(t)$ be a geodesic. Then $\|x'(t)\|$ is constant.*

ii. [3]*Any (part of a) straight line on a surface is a geodesic.*

iii. [4]*Any normal section of a surface is a geodesic.*

Proof. Left as exercises. □

2.2. Geodesic equations

- [5]A curve $x(t) = \sigma(u(t), v(t))$ is a geodesic $\Longleftrightarrow$

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathbb{E}\,u' + \mathbb{F}\,v') = \frac{1}{2}\,(\mathbb{E}_u\,(u')^2 + 2\,\mathbb{F}_u\,u'\,v' + \mathbb{G}_u\,(v')^2), \tag{44}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathbb{F}\,u' + \mathbb{G}\,v') = \frac{1}{2}\,(\mathbb{E}_v\,(u')^2 + 2\,\mathbb{F}_v\,u'\,v' + \mathbb{G}_v\,(v')^2). \tag{45}$$

2. Proposition 9.1.2 of the textbook.
3. Proposition 9.1.4 of the textbook.
4. Proposition 9.1.6 of the textbook.
5. Theorem 9.2.1 of the textbook.

(44–45) are called *geodesic equations*.

Proof. (44–45) is equivalent to $\kappa = |\kappa_n|$ everywhere along the curve. See Homework 5 for details. □

Remark 27. [6]Any local isometry between two surfaces takes the geodesics of one surface to the geodesics of the other.

Remark 28. It is useful to notice that

$$\mathbb{E}\, u' + \mathbb{F}\, v' = x'(t) \cdot \sigma_u, \qquad \mathbb{F}\, u' + \mathbb{G}\, v' = x'(t) \cdot \sigma_v \tag{46}$$

and the right hand side takes simpler form in matrix form:

$$\frac{1}{2} \begin{pmatrix} u' & v' \end{pmatrix} \begin{pmatrix} \mathbb{E} & \mathbb{F} \\ \mathbb{F} & \mathbb{G} \end{pmatrix}_u \begin{pmatrix} u' \\ v' \end{pmatrix}, \qquad \frac{1}{2} \begin{pmatrix} u' & v' \end{pmatrix} \begin{pmatrix} \mathbb{E} & \mathbb{F} \\ \mathbb{F} & \mathbb{G} \end{pmatrix}_v \begin{pmatrix} u' \\ v' \end{pmatrix}. \tag{47}$$

- [7]Equations (44–45) can be re-written as

$$u'' + \Gamma_{11}^1 (u')^2 + 2\, \Gamma_{12}^1 u' v' + \Gamma_{22}^1 (v')^2 = 0, \tag{48}$$
$$v'' + \Gamma_{11}^2 (u')^2 + 2\, \Gamma_{12}^2 u' v' + \Gamma_{22}^2 (v')^2 = 0. \tag{49}$$

Proof. This follows immediately from Theorem 15 once we set $w(t) = x'(t) = u'(t)\, \sigma_u + v'(t)\, \sigma_v$. □

Remark 29. (48) and (49) take simple matrix forms:

$$u'' + \begin{pmatrix} u' & v' \end{pmatrix} \begin{pmatrix} \Gamma_{11}^1 & \Gamma_{12}^1 \\ \Gamma_{12}^1 & \Gamma_{22}^1 \end{pmatrix} \begin{pmatrix} u' \\ v' \end{pmatrix} = 0, \tag{50}$$

and

$$v'' + \begin{pmatrix} u' & v' \end{pmatrix} \begin{pmatrix} \Gamma_{11}^2 & \Gamma_{12}^2 \\ \Gamma_{12}^2 & \Gamma_{22}^2 \end{pmatrix} \begin{pmatrix} u' \\ v' \end{pmatrix} = 0. \tag{51}$$

The pattern would be crystal clear if we replace u by u_1 and v by u_2.

- Examples.

Example 30. [8]We consider a surface of revolution

$$\sigma(u, v) = (f(u) \cos v, f(u) \sin v, u). \tag{52}$$

We have

$$\sigma_u = (f'(u) \cos v, f'(u) \sin v, 1), \qquad \sigma_v = (-f(u) \sin v, f(u) \cos v, 0) \tag{53}$$

6. Corollary 9.2.7 of the textbook.

7. Proposition 9.2.3 of the textbook.

8. Proposition 9.3.1, 9.3.2 of the textbook.

which leads to

$$\mathbb{E} = 1 + f'(u)^2, \qquad \mathbb{F} = 0, \qquad \mathbb{G} = f(u)^2. \tag{54}$$

Thus (44–45) becomes

$$((1 + f'(u)^2)\, u')' = f'(u)\, f''(u)\, (u')^2 + f(u)\, f'(u)\, (v')^2, \tag{55}$$

$$(f(u)^2\, v')' = 0. \tag{56}$$

We see that $f(u)^2\, v' = C$ is a constant. The first equation simplifies to

$$(1 + f'(u)^2)\, u'' = f(u)\, f'(u)\, (v')^2. \tag{57}$$

We see that

- $v =$ constant are geodesics;
- $u = u_0$ where $f'(u_0) = 0$ are geodesics.
- We see that

$$\cos\angle(\sigma_u, x') = \frac{\sqrt{1 + f'(u)^2}\, u'}{\sqrt{(1 + f'(u)^2)\,(u')^2 + f(u)^2\,(v')^2}} \tag{58}$$

which gives

$$\sin^2\angle(\sigma_u, x') = \frac{f(u)^2\,(v')^2}{(1 + f'(u)^2)\,(u')^2 + f(u)^2\,(v')^2}. \tag{59}$$

Now notice that, as $x(t)$ is geodesic, $(1 + f'(u)^2)\,(u')^2 + f(u)^2\,(v')^2 = \|x'(t)\|^2 =$ constant. Consequently we have

$$f(u)^2 \sin^2\angle(\sigma_u, x') = \text{constant} \tag{60}$$

which is (almost) Clairaut's theorem.

Exercise 6. Prove this directly from (55),(56).

Example 31. Let S be the unit sphere. This of course is a surface of revolution. We re-write it as

$$(f(u)\cos v, f(u)\sin v, u) \tag{61}$$

where $f(u) = \cos(\arcsin u) = \sqrt{1 - u^2}$. By Example 30 we see that for any geodesic $x(s) = (f(u(s))\cos v(s), f(u(s))\sin v(s), u(s))$, parametrized by arc length, there holds

$$(1 - u^2)\, v' = f(u)^2\, v' = \text{constant} =: c_0. \tag{62}$$

On the other hand, as $\|x'(s)\| = 1$ we have

$$1 = f'(u)^2\,(u')^2 + f(u)^2\,(v')^2 + (u')^2 = \frac{(u')^2}{1 - u^2} + (1 - u^2)\,(v')^2. \tag{63}$$

Therefore

$$1 - c_0^2 = u^2 + (u')^2. \tag{64}$$

Now we have (we simply write u for $u(s)$)

$$x(s) = \left(\sqrt{1-u^2}\cos v, \sqrt{1-u^2}\sin v, u\right) \tag{65}$$

and

$$x'(s) = \left(\frac{-u\,u'}{\sqrt{1-u^2}}\cos v - \sqrt{1-u^2}\sin v\, v', \frac{-u\,u'}{\sqrt{1-u^2}}\sin v + \sqrt{1-u^2}\cos v\, v', u'\right). \tag{66}$$

Consequently

$$x(s) \times x'(s) = \left(\frac{u'\sin v - c_0\, u\cos v}{\sqrt{1-u^2}}, -\frac{u'\cos v + c_0\, u\sin v}{\sqrt{1-u^2}}, c_0\right) \tag{67}$$

Now calculate (using (62) and (64) whenever applicable)

$$\begin{aligned}\left(\frac{u'\sin v - c_0\, u\cos v}{\sqrt{1-u^2}}\right)' &= \frac{u''\sin v + u'\cos v\, v' - c_0\, u'\cos v + c_0\, u\sin v\, v'}{\sqrt{1-u^2}} \\ &\quad + \frac{u\,u'\,(u'\sin v - c_0\, u\cos v)}{\sqrt{1-u^2}^3} \\ &= \frac{u''\,(1-u^2)\sin v + (c_0^2 + (u')^2)u\sin v}{\sqrt{1-u^2}^3} \\ &= \frac{u''\,(1-u^2)\sin v + (1-u^2)\,u\sin v}{\sqrt{1-u^2}^3}.\end{aligned} \tag{68}$$

Finally notice that differentiating (64) gives

$$(u'' + u)\,u' = 0. \tag{69}$$

Thus if $u' \neq 0$, there holds $u'' = -u$ and consequently $\left(\frac{u'\sin v - c_0\, u\cos v}{\sqrt{1-u^2}}\right)' = 0$. Similarly $\left(-\frac{u'\cos v + c_0\, u\sin v}{\sqrt{1-u^2}}\right)' = 0$ and $x(s) \times x'(s)$ is a constant vector. This implies $x(s)$ lies in a plane passing the origin and must be part of a big circle.

Exercise 7. Rigorously prove this last claim: Let $x(s)$ be a curve on the unit sphere parametrized by arc length. Assume that $x(s) \times x'(s)$ is a constant vector. Then $x(s)$ lies in a plane passing the origin.

Exercise 8. What if $u' = 0$?

2.3. Geodesics as shortest paths

- On the flat plane, the shortest path connecting any two points is the one that is part of a geodesic, which is a straight line.
- Set up. Let $p_1, p_2 \in S$ and let γ_0 be the shortest path connecting p_1, p_2. Parametrize γ_0 by arc length $\sigma(u_0(s), v_0(s))$. Set $x(s_1) = p_1, x(s_2) = p_2$. Now let $(u_1(s), v_1(s))$ be an arbitrary vector field along $(u_0(s), v_0(s))$ in $\mathbb{R}^2$ and $\lambda(s)$: $\mathbb{R} \mapsto \mathbb{R}$ be such that $\lambda(s_1) = \lambda(s_2) = 0$. Let $\tau \in \mathbb{R}$ and denote by γ_τ the curve $\sigma(u_0 + \tau\,\lambda\, u_1, v_0 + \tau\lambda\, v_1)$. Finally denote

$$L(\tau) = \int_{s_1}^{s_2} \left\| \frac{\mathrm{d}}{\mathrm{d}s}\sigma(u_0(s) + \tau\lambda(s)\, u_1(s), v_0(s) + \tau\lambda(s)\, v_1(s)) \right\| \mathrm{d}s. \tag{70}$$

- Calculus of variations. We have

$$\begin{aligned} L(\tau) &= \int_{s_1}^{s_2} \|(u_0+\tau\lambda u_1)'\sigma_u+(v_0+\tau\lambda v_1)'\sigma_v\|\,\mathrm{d}s \\ &= \int_{s_1}^{s_2} [((u_0+\tau\lambda u_1)'\sigma_u+(v_0+\tau\lambda v_1)'\sigma_v)\cdot((u_0+\tau\lambda u_1)'\sigma_u+(v_0+\\ &\quad \tau\lambda v_1)'\sigma_v)]^{1/2}\,\mathrm{d}s. \end{aligned} \tag{71}$$

Here it is crucial to realize that σ_u, σ_v are evaluated at $(u_0+\tau\lambda u_1, v_0+\tau\lambda v_1)$. In particular, they are dependent on τ.

Now we calculate

$$\begin{aligned} L'(0) &= \int_{s_1}^{s_2} (\mathbb{E}_0\, u_0'^2 + 2\,\mathbb{F}_0\, u_0'\, v_0' + \mathbb{G}_0\, v_0'^2)^{-1/2}\, [(u_0'\,\sigma_u + v_0'\,\sigma_v)\cdot((\lambda\, u_1)'\,\sigma_u + \\ &\quad (\lambda v_1)'\sigma_v + u_0'(\lambda u_1\sigma_{uu}+\lambda v_1\sigma_{uv}) + v_0'(\lambda u_1\sigma_{vu}+\lambda v_1\sigma_{vv}))]\,\mathrm{d}s \\ &= \int_{s_1}^{s_2} (\lambda u_1)'[(u_0'\sigma_u+v_0'\sigma_v)\cdot\sigma_u] + (\lambda v_1)'[(u_0'\sigma_u+v_0'\sigma_v)\cdot\sigma_v]\,\mathrm{d}s \\ &\quad + \int_{s_1}^{s_2} (\lambda u_1)\,[(u_0'\sigma_{uu}+v_0'\sigma_{uv})\cdot(u_0'\sigma_u+v_0'\sigma_v)]\,\mathrm{d}s \\ &\quad + \int_{s_1}^{s_2} (\lambda v_1)\,[(u_0'\sigma_{uv}+v_0'\sigma_{vv})\cdot(u_0'\sigma_u+v_0'\sigma_v)]\,\mathrm{d}s. \end{aligned} \tag{72}$$

Note that in the above we have used the fact that $\mathbb{E}_0\, u_0'^2 + 2\,\mathbb{F}_0\, u_0'\, v_0' + \mathbb{G}_0\, v_0'^2 = \|x'(s)\|^2 = 1$. Also note that in (72) $\sigma_u, \ldots, \sigma_{vv}$ are all evaluated at (u_0, v_0) now.

Next we integrate the first integral in (72) by parts and collect all the u_1 terms together, and all the v_1 terms together.

$$\begin{aligned} L'(0) &= -\int_{s_1}^{s_2} (\lambda u_1)\,[((u_0'\sigma_u+v_0'\sigma_v)\cdot\sigma_u)' - (u_0'\sigma_{uu}+v_0'\sigma_{uv})\cdot(u_0'\sigma_u+v_0'\sigma_v)]\,\mathrm{d}s \\ &\quad -\int_{s_1}^{s_2} (\lambda v_1)\,[((u_0'\sigma_u+v_0'\sigma_v)\cdot\sigma_v)' - (u_0'\sigma_{uv}+v_0'\sigma_{vv})\cdot(u_0'\sigma_u+v_0'\sigma_v)]\,\mathrm{d}s. \end{aligned}$$

Due to the arbitrariness of u_1, v_1, we conclude

$$((u_0'\sigma_u+v_0'\sigma_v)\cdot\sigma_u)' = (u_0'\sigma_{uu}+v_0'\sigma_{uv})\cdot(u_0'\sigma_u+v_0'\sigma_v), \tag{73}$$

$$((u_0'\sigma_u+v_0'\sigma_v)\cdot\sigma_v)' = (u_0'\sigma_{uv}+v_0'\sigma_{vv})\cdot(u_0'\sigma_u+v_0'\sigma_v). \tag{74}$$

Simple calculation now gives

$$(\mathbb{E}\,u_0'+\mathbb{F}\,v_0')' = \frac{1}{2}(\mathbb{E}_u\,(u_0')^2 + 2\,\mathbb{F}_u\,u_0'\,v_0' + \mathbb{G}_u\,(v_0')^2), \tag{75}$$

$$(\mathbb{F}\,u_0'+\mathbb{G}\,v_0') = \frac{1}{2}(\mathbb{E}_v\,(u_0')^2 + 2\,\mathbb{F}_v\,u_0'\,v_0' + \mathbb{G}_v\,(v_0')^2). \tag{76}$$

Exercise 9. Derive (75–76) from (73–74).

Remark 32. Note that a shortest path must be a geodesic but a geodesic does not necessarily give shortest path.