Lectures 10–11: How Does a Surface Curve

Disclaimer. As we have a textbook, this lecture note is for guidance and supplement only. It should not be relied on when preparing for exams.

In this lecture we study how to measure the curving of a surface patch. The required textbook sections are §7.1–7.3.

I try my best to make the examples in this note different from examples in the textbook. Please read the textbook carefully and try your hands on the exercises. During this please don't hesitate to contact me if you have any questions.

Table of contents

Lectures 10–11: How Does a Surface Curve 1

1. Distance to the tangent plane 2
2. The turning of the unit normal 2
3. How much are the curves in the surface curving? 5
4. The second fundament form 8
   - 4.1. Definition 8
   - 4.2. Properties 9
5. Examples 10
   - 5.1. Calculation of the second fundamental form 10
   - 5.2. Applications of the second fundamental form 11

Let S be a surface and let $p_0 \in S$. Let $\sigma: U \mapsto \mathbb{R}^3$ be a surface patch covering p_0. Let $\sigma(u_0, v_0) = p_0$. In the following we study three ways to measure how the surface curves at p_0.

1. Distance to the tangent plane

- We measure the curving of the surface by calculating how quickly the surface curves away from its tangent plane at p_0. Note that the tangent plane is the best flat approximation of the surface that passes p_0.
- Recall that the equation for the tangent plane in $\mathbb{R}^3$ is given by

$$(x - p_0) \cdot N(p_0) = 0. \tag{1}$$

- Let $p = \sigma(u, v) \in S$ be a point close to p. Then we have its distance to the tangent plane to be

$$d(u, v) = |(\sigma(u, v) - \sigma(u_0, v_0)) \cdot N(\sigma(u_0, v_0))|. \tag{2}$$

- We calculate $d(u, v)$ through Taylor expansion:

$$\begin{aligned}(\sigma(u, v) - \sigma(u_0, v_0)) \cdot N(\sigma(u_0, v_0)) &= [\sigma_u (u - u_0) + \sigma_v (v - v_0)] \cdot N \\ &\quad + \left[\frac{1}{2} \sigma_{uu} (u - u_0)^2 + \sigma_{uv} (u - u_0) (v - v_0) + \frac{1}{2} \sigma_{vv} (v - v_0)^2\right] \cdot N + R(u, v) \cdot N \\ &= \frac{1}{2} [\mathbb{L} (u - u_0)^2 + 2 \mathbb{M} (u - u_0) (v - v_0) + \mathbb{N} (v - v_0)^2] + R(u, v) \cdot N, \end{aligned} \tag{3}$$

  where $\lim_{(u,v) \longrightarrow (u_0, v_0)} \frac{|R(u, v)|}{(u - u_0)^2 + (v - v_0)^2} = 0.$

- Thus we see that the curving of the surface at p_0 can be characterized by three numbers:

$$\mathbb{L}(u_0, v_0) := \sigma_{uu}(u_0, v_0) \cdot N(u_0, v_0), \tag{4}$$
$$\mathbb{M}(u_0, v_0) := \sigma_{uv}(u_0, v_0) \cdot N(u_0, v_0), \tag{5}$$
$$\mathbb{N}(u_0, v_0) := \sigma_{vv}(u_0, v_0) \cdot N(u_0, v_0). \tag{6}$$

Exercise 1. Would we obtain the same numbers if we use $N(\sigma(u, v))$ instead of $N(\sigma(u_0, v_0))$ in (2)?

2. The turning of the unit normal

- Recall that the unit normal vector $N(p) := \frac{\sigma_u \times \sigma_v}{\|\sigma_u \times \sigma_v\|}$ can be thought of as a mapping from S to the unit sphere $\mathbb{S}^2$. This map is called the *Gauss map* and will be denote by $\mathcal{G}$.

Notation Change!

From now on we will use $\mathcal{G}$ to denote the Gauss map from a point $p \in S$ to the unit normal there, and will use the old notation N in the following way: $N(u,v) := \mathcal{G}(\sigma(u,v))$, that is $N := \mathcal{G} \circ \sigma$.

- The curving of S at p_0 should be characterized by the differential $D_{p_0}\mathcal{G}$. Recall that for a velocity $w \in T_{p_0}S$, $D_{p_0}\mathcal{G}(w)$ is the angular velocity of the turning of the unit normal.

Definition 1. (Definition 7.2.1 in the textbook) *We define the* Weingarten map

$$\mathcal{W}_{p_0,S} := -D_{p_0}\mathcal{G} \tag{7}$$

where $\mathcal{G}$ is the Gauss map.

Note the minus sign here.

Example 2. We try to calculate $\mathcal{W}_{p_0,S}(\sigma_u)$ and $\mathcal{W}_{p_0,S}(\sigma_v)$ for the following surface patches. It is clear that

$$\mathcal{W}_{p_0,S}(\sigma_u) = -N_u, \qquad \mathcal{W}_{p_0,S}(\sigma_v) = -N_v. \tag{8}$$

a) S is the plane $\sigma(u,v) = (u, v, 3u + 2v)$.

In this case we have

$$\sigma_u = (1, 0, 3), \qquad \sigma_v = (0, 1, 2) \tag{9}$$

which give

$$N(u,v) = \mathcal{G}(\sigma(u,v)) = \frac{\sigma_u \times \sigma_v}{\|\sigma_u \times \sigma_v\|} = \frac{1}{\sqrt{14}}(-3, -2, 1). \tag{10}$$

We see that $\mathcal{W}(\sigma_u) = \mathcal{W}(\sigma_v) = 0$.

b) S is the cylinder $\sigma(u,v) = (\cos u, \sin u, v)$.

In this case we have

$$\sigma_u = (-\sin u, \cos u, 0), \qquad \sigma_v = (0, 0, 1) \tag{11}$$

and

$$N(u,v) = \frac{\sigma_u \times \sigma_v}{\|\sigma_u \times \sigma_v\|} = (\cos u, \sin u, 0). \tag{12}$$

We have

$$N_u = (-\sin u, \cos u, 0) = \sigma_u, \qquad N_v = (0, 0, 0). \tag{13}$$

Consequently we have

$$\mathcal{W}(\sigma_u) = -\sigma_u, \qquad \mathcal{W}(\sigma_v) = 0. \tag{14}$$

c) S is the unit sphere $\sigma(u,v) = \left(u, v, \sqrt{1 - u^2 - v^2}\right)$.

We have

$$\sigma_u = \left(1, 0, \frac{-u}{\sqrt{1 - u^2 - v^2}}\right), \qquad \sigma_v = \left(0, 1, \frac{-v}{\sqrt{1 - u^2 - v^2}}\right) \tag{15}$$

and

$$N(u,v)=\left(u,v,\sqrt{1-u^2-v^2}\right)=\sigma(u,v). \tag{16}$$

Consequently

$$\mathcal{W}(\sigma_u)=-N_u, \qquad \mathcal{W}(\sigma_v)=-N_v. \tag{17}$$

d) S is the hyperbolic paraboloid $\sigma(u,v)=(u,v,u\,v)$ with $p_0=(0,0,0)$.

We have

$$\sigma_u=(1,0,v), \qquad \sigma_v=(0,1,u) \tag{18}$$

and

$$N(u,v)=\left(\frac{-v}{\sqrt{1+u^2+v^2}},\frac{-u}{\sqrt{1+u^2+v^2}},\frac{1}{\sqrt{1+u^2+v^2}}\right). \tag{19}$$

Now we calculate

$$\mathcal{W}(\sigma_u)=-N_u=\left(\frac{-u\,v}{(1+u^2+v^2)^{3/2}},\frac{1+v^2}{(1+u^2+v^2)^{3/2}},\frac{u}{(1+u^2+v^2)^{3/2}}\right) \tag{20}$$

and

$$\mathcal{W}(\sigma_v)=-N_v=\left(\frac{1+u^2}{(1+u^2+v^2)^{3/2}},\frac{-u\,v}{(1+u^2+v^2)^{3/2}},\frac{v}{(1+u^2+v^2)^{3/2}}\right). \tag{21}$$

We see that

$$\mathcal{W}(\sigma_u)=-\frac{u\,v}{(1+u^2+v^2)^{3/2}}\,\sigma_u+\frac{1+v^2}{(1+u^2+v^2)^{3/2}}\,\sigma_v \tag{22}$$

and

$$\mathcal{W}(\sigma_v)=\frac{1+u^2}{(1+u^2+v^2)^{3/2}}\,\sigma_u-\frac{u\,v}{(1+u^2+v^2)^{3/2}}\,\sigma_v. \tag{23}$$

Exercise 2. Try to interpret the above calculation results. What exactly does $\mathcal{W}$ do in each case?

- **Failed attempts to understand the Weingarten map.** Naturally we would like to calculate the matrix representation of $\mathcal{W}_{p_0,S}$. Let $\tilde{\sigma}\colon \tilde{U}\mapsto \mathbb{S}^2$ be a surface patch of $\mathbb{S}^2$ covering $N(p_0)$. Then we have

$$F(u,v)=\tilde{\sigma}^{-1}\circ(-\mathcal{G})\circ\sigma=\tilde{\sigma}^{-1}\left(-\frac{\sigma_u\times\sigma_v}{\|\sigma_u\times\sigma_v\|}\right). \tag{24}$$

Consequently

$$DF(u,v)=-D(\tilde{\sigma}^{-1})\cdot\left(\ \left(\frac{\sigma_u\times\sigma_v}{\|\sigma_u\times\sigma_v\|}\right)_u\ \left(\frac{\sigma_u\times\sigma_v}{\|\sigma_u\times\sigma_v\|}\right)_v\ \right) \tag{25}$$

where $\left(\frac{\sigma_u\times\sigma_v}{\|\sigma_u\times\sigma_v\|}\right)_u, \left(\frac{\sigma_u\times\sigma_v}{\|\sigma_u\times\sigma_v\|}\right)_v$ are written as column vectors.

Exercise 3. Try to carry out the calculation.

Exercise 4. Try to instead calculate the first fundamental form of the sphere $\mathbb{S}^2$ by the surface patch $\mathcal{G}(\sigma(u,v))$. Note that this first fundamental form is also called the third fundamental form of S.

- The key observation.

Remark 3. There is indeed one particular surface patch $\tilde{\sigma}$ which allows us to easily calculate the matrix representation of $D_p\mathcal{G}$. However this matrix representation is useless.

Exercise 5. What is this matrix representation if we take $\tilde{\sigma} = N$? Why is it useless?

We have seen that $\mathcal{W}(\sigma_u) = -N_u$, $\mathcal{W}(\sigma_v) = -N_v$. As $\mathcal{W}$ is linear, for $a, b \in \mathbb{R}$ we have

$$\mathcal{W}(a\,\sigma_u + b\,\sigma_v) = -a\,N_u - b\,N_v. \tag{26}$$

Therefore to understand $\mathcal{W}$ we need to understand N_u, N_v. The crucial observation is the following.

$$N_u, N_v \perp N \Longrightarrow -N_u = a_{11}\,\sigma_u + a_{12}\,\sigma_v, -N_v = a_{21}\,\sigma_u + a_{22}\,\sigma_v.$$

- Calculating $a_{11}, \ldots, a_{22}$.

Theorem 4. *We have*

$$\begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix} = \begin{pmatrix} \mathbb{E} & \mathbb{F} \\ \mathbb{F} & \mathbb{G} \end{pmatrix}^{-1} \begin{pmatrix} \mathbb{L} & \mathbb{M} \\ \mathbb{M} & \mathbb{N} \end{pmatrix} \tag{27}$$

where $\mathbb{E}\,\mathrm{d}u^2 + 2\,\mathbb{F}\,\mathrm{d}u\,\mathrm{d}v + \mathbb{G}\,\mathrm{d}v^2$ *is the first fundamental form of* S *at* p_0*, and* $\mathbb{L}, \mathbb{M}$, $\mathbb{N}$ *are defined in (4–6).*

Proof. We notice that as $\sigma_u \cdot N = \sigma_v \cdot N = 0$, there holds

$$\mathbb{L} = \sigma_{uu} \cdot N = (\sigma_u \cdot N)_u - \sigma_u \cdot N_u = -\sigma_u \cdot N_u \tag{28}$$

and similarly

$$\mathbb{M} = -\sigma_v \cdot N_u = -\sigma_u \cdot N_v, \qquad \mathbb{N} = -\sigma_v \cdot N_v. \tag{29}$$

This leads to

$$\mathbb{E}\,a_{11} + \mathbb{F}\,a_{12} = \sigma_u \cdot (a_{11}\,\sigma_u + a_{12}\,\sigma_v) = -\sigma_u \cdot N_u = \mathbb{L}, \tag{30}$$

$$\mathbb{F}\,a_{11} + \mathbb{G}\,a_{12} = \sigma_v \cdot (a_{11}\,\sigma_u + a_{12}\,\sigma_v) = -\sigma_v \cdot N_u = \mathbb{M}. \tag{31}$$

Consequently

$$\begin{pmatrix} a_{11} \\ a_{12} \end{pmatrix} = \begin{pmatrix} \mathbb{E} & \mathbb{F} \\ \mathbb{F} & \mathbb{G} \end{pmatrix}^{-1} \begin{pmatrix} \mathbb{L} \\ \mathbb{M} \end{pmatrix}. \tag{32}$$

Similarly we have $\begin{pmatrix} a_{21} \\ a_{22} \end{pmatrix} = \begin{pmatrix} \mathbb{E} & \mathbb{F} \\ \mathbb{F} & \mathbb{G} \end{pmatrix}^{-1} \begin{pmatrix} \mathbb{M} \\ \mathbb{N} \end{pmatrix}$ and the conclusion follows. □

3. How much are the curves in the surface curving?

- Let $x(t) := \sigma(u(t), v(t))$ be a curve in S with $u(t_0) = u_0$, $v(t_0) = v_0$. Thus it passes $p_0 = \sigma(u_0, v_0)$. We try to understand the curving of S at p_0 through the curvature of $x(t)$ at $x(t_0)$.

- To make this idea work we need to first qualitatively understand how are the curving of S at p and the curvature of $x(t)$ related.

  Example 5. We consider the following paradigm situations.

  - Let S be the plane and $p_0 \in S$. Clearly a curve passing p_0 can have any curvature.
  - Let S be the cylinder and $p_0 \in S$. Again a curve passing p_0 can have arbitrary $\kappa_0 \geqslant 0$ as its curvature there.
  - Let S be the unit sphere. Intuitively we see that a curve passing $p_0 \in S$ could have any curvature $\geqslant 1$ but not <1.

    Exercise 6. Prove this.

  From these examples it seems that the relations between the curvature of $x(t)$ and the curving S is very loose. However, this relation becomes much more precise when we consider not all possible curvatures, but the minimal one:

  > Given any unit vector $w \in T_{p_0}S$, let $\kappa_{\min}(w)$ be the minimal curvature of all possible curvatures of the curves passing p_0 and are tangent to w at p_0.

  Now we see that $\kappa_{\min}$ very precisely reflects the curving of the surface.

  - For S the flat plane: $\kappa_{\min}(w) = 0$ for all w;
  - For S the cylinder: $\kappa_{\min}(w) = 0$ when $w = (0, 0, 1)$ and $\kappa_n(w) = 1$ when w is the horizontal tangent, and $\kappa_{\min}(w)$ lies between 0 and 1 for other directions.
  - For S the sphere: $\kappa_{\min}(w) = 1$ for all w.

- What is $\kappa_{\min}(w)$?

  First we re-parametrize by arc length $x(s) = \sigma(u(s), v(s))$. We calculate

$$x'(s) = u'(s)\,\sigma_u + v'(s)\,\sigma_v, \tag{33}$$

$$x''(s) = u''(s)\,\sigma_u + v''(s)\,\sigma_v + u'(s)^2\,\sigma_{uu} + 2\,u'(s)\,v'(s)\,\sigma_{uv} + v'(s)^2\,\sigma_{vv}. \tag{34}$$

  Let T, N be the unit tangent and normal of the curve $x(s)$ at $x(s_0) = p_0$, and denote by $N_S := \frac{\sigma_u \times \sigma_v}{\|\sigma_u \times \sigma_v\|}$ the unit normal at $p_0 = \sigma(u_0, v_0)$. As we require $x(s)$ to be tangent to a fixed direction, $u'(s_0), v'(s_0)$ are fixed. Therefore we further denote

$$u_1 := u'(s_0), \qquad v_1 := v'(s_0) \tag{35}$$

  to emphasize this point. Thus we have

$$x''(s_0) = u''(s_0)\,\sigma_u + v''(s_0)\,\sigma_v + u_1^2\,\sigma_{uu} + 2\,u_1\,v_1\,\sigma_{uv} + v_1^2\,\sigma_{vv}. \tag{36}$$

  Next observe that $N \parallel x''(s_0) \perp T,\ T \perp N_S$. We see that

$$\kappa \geqslant |x''(s_0) \cdot N_S| = |\mathbb{L}\,u_1^2 + 2\,\mathbb{M}\,u_1\,v_1 + \mathbb{N}\,v_1^2| \tag{37}$$

  thanks to the fact that $\sigma_u \cdot N_S = \sigma_v \cdot N_S = 0$.

As σ_u, σ_v form a basis of $T_{p_0}S$, it is always possible to find $u''(s_0), v''(s_0)$ such that $x''(s_0) \parallel N_S$. Consequently, we conclude (when $\|u_1\,\sigma_u + v_1\,\sigma_v\| = 1$)

$$\kappa_{\min}(u_1\,\sigma_u + v_1\,\sigma_v) = |\mathbb{L}\,u_1^2 + 2\,\mathbb{M}\,u_1\,v_1 + \mathbb{N}\,v_1^2|. \tag{38}$$

Remark 6. A curve $x(t) = \sigma(u(t), v(t))$ satisfy $\kappa(t) = |\kappa_{\min}(T(t))|$ at every t if and only if $u(t), v(t)$ satisfy the following equations

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathbb{E}\,u' + \mathbb{F}\,v') = \frac{1}{2}\,(\mathbb{E}_u\,(u')^2 + 2\,\mathbb{F}_u\,u'\,v' + \mathbb{G}_u\,(v')^2), \tag{39}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathbb{F}\,u' + \mathbb{G}\,v') = \frac{1}{2}\,(\mathbb{E}_v\,(u')^2 + 2\,\mathbb{F}_v\,u'\,v' + \mathbb{G}_v\,(v')^2). \tag{40}$$

Exercise 7. Prove this.

- Normal and geodesic curvatures.

DEFINITION 7. *Let $x(t) := \sigma(u(t), v(t))$ be a curve in S passing $p_0 = \sigma(u(t_0), v(t_0))$. Denote by T, N the unit tangent direction and unit normal direction of $x(t)$ at p_0, and by N_S the unit normal direction of S at p_0. Denote by κ the curvature of $x(t)$ at p_0. Then*

$$\kappa\,N = \kappa_n\,N_S + \kappa_g\,(N_S \times T). \tag{41}$$

*We call κ_n the **normal curvature** and κ_g the **geodesic curvature** of $x(t)$ at p_0.*

- Properties.
  - There holds
  $$\kappa^2 = \kappa_n^2 + \kappa_g^2. \tag{42}$$
  - $|\kappa_n|$ is the smallest possible curvature for all curves in S passing p_0 with $x'(t)$ parallel to the fixed direction $w \in T_p(S)$.
  - Let $w \in T_pS$ be fixed. Let $x(t)$ be the intersection of S with the plane passing p_0 spanned by w and N_S[1]. Then the curvature of $x(t)$ at p_0 is $|\kappa_n|$.

Warning

The curvature of $x(t)$ at $p \neq p_0$ may not equal to $|\kappa_n(p)|$ anymore.

Exercise 8. Find an example illustrating this. (One possibility is cylinder).

  - In general, we have
  $$\kappa_n = \kappa\cos\psi, \quad \kappa_g = \pm\kappa\sin\psi \tag{43}$$
  where ψ is the angle between N_S and N.

  In particular, if $x(t)$ is the intersection of S with a plane passing the line through p_0 in the direction w, then the curvature of $x(t)$ at p_0 is given by
  $$\kappa = \frac{|\kappa_n|}{\cos\psi} \tag{44}$$

1. Such $x(t)$ is called a "normal section"

where ψ is the angle between the plane and the unit normal N_S to the surface at p_0.

4. The second fundament form

4.1. Definition

First we summarize our three approaches.

1. The distance of a point $\sigma(u,v)$ to the tangent plane at $p_0=\sigma(u_0,v_0)$ is $\frac{1}{2}\,[\mathbb{L}\,(u-u_0)^2+2\,\mathbb{M}\,(u-u_0)\,(v-v_0)+\mathbb{N}\,(v-v_0)^2]$;
2. The Weingarten map $\mathcal{W}=-D_{p_0}\mathcal{G}$ is given by
$$\mathcal{W}(\lambda\,\sigma_u+\mu\,\sigma_v)=\lambda\,(a_{11}\,\sigma_u+a_{12}\,\sigma_v)+\mu\,(a_{21}\,\sigma_u+a_{22}\,\sigma_v) \tag{45}$$
where
$$\begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix}=\begin{pmatrix} \mathbb{E} & \mathbb{F} \\ \mathbb{F} & \mathbb{G} \end{pmatrix}^{-1}\begin{pmatrix} \mathbb{L} & \mathbb{M} \\ \mathbb{M} & \mathbb{N} \end{pmatrix} \tag{46}$$
3. At p_0, if we fix a unit vector $w:=u_1\,\sigma_u+v_1\,\sigma_v\in T_{p_0}S$, and consider all curves $x(t)$ satisfying $x(t_0)=p_0$, $x'(t_0)\parallel w$, then there holds
$$\kappa(t_0)\geqslant|\kappa_n(w)| \tag{47}$$
where
$$\kappa_n(w):=\mathbb{L}\,u_1^2+2\,\mathbb{M}\,u_1\,v_1+\mathbb{N}\,v_1 \tag{48}$$
is called the *normal curvature of S at p_0 in the direction w*. One particular curve among those satisfying $x(t_0)=p_0$, $x'(t_0)\parallel w$ with $\kappa(t_0)=|\kappa_n(w)|$ is the curve obtained as the intersection between S and the plane passing p_0 spanned by N_S and w.

We see that the three numbers (functions if we consider all $p\in S$) $\mathbb{L}$, $\mathbb{M}$, $\mathbb{N}$ plays a crucial role in determining how much a surface curves. This inspires the following definition.

Definition 8. (The second fundamental form) *Let S be a surface and $p_0\in S$. Let σ be a surface patch of S covering p_0: $p_0=\sigma(u_0,v_0)$. Then the second fundamental form of S at p_0, denoted $\langle\langle\cdot,\rangle\rangle_{p_0,S}$ (with p, S omitted when no confusion may arise), is a bilinear form on $T_{p_0}S$ defined through*
$$\mathbb{L}(u_0,v_0)\,\mathrm{d}u^2+2\,\mathbb{M}(u_0,v_0)\,\mathrm{d}u\,\mathrm{d}v+\mathbb{N}(u_0,v_0)\,\mathrm{d}v^2 \tag{49}$$
where
$$\begin{aligned}\mathbb{L}(u_0,v_0) &:= \sigma_{uu}(u_0,v_0)\cdot N(u_0,v_0)=-\sigma_u\cdot N_u, &(50)\\ \mathbb{M}(u_0,v_0) &:= \sigma_{uv}(u_0,v_0)\cdot N(u_0,v_0)=-\sigma_u\cdot N_v=-\sigma_v\cdot N_u, &(51)\\ \mathbb{N}(u_0,v_0) &:= \sigma_{vv}(u_0,v_0)\cdot N(u_0,v_0)=-\sigma_v\cdot N_v. &(52)\end{aligned}$$

Remark 9. If $w=w_1\,\sigma_u+w_2\,\sigma_v$ and $\tilde{w}=\tilde{w}_1\,\sigma_u+\tilde{w}_2\,\sigma_v$, then we have
$$\langle\langle w,\tilde{w}\rangle\rangle=\mathbb{L}\,w_1\,\tilde{w}_1+\mathbb{M}\,(w_1\,\tilde{w}_2+w_2\,\tilde{w}_1)+\mathbb{N}\,w_2\,\tilde{w}_2. \tag{53}$$

Remark 10. Let $x(t) = \sigma(u(t), v(t))$. We clearly have

$$\kappa_n = \mathbb{L}\,(u')^2 + 2\,\mathbb{M}\,u'\,v' + \mathbb{N}\,(v')^2 = \langle\langle x', x'\rangle\rangle_{x(t),S} \tag{54}$$

when $x(t)$ is parametrized by arc length. We can further prove the following general formula.

$$\kappa_n(p) = \frac{\langle\langle\cdot,\cdot\rangle\rangle_{p,S}}{\langle\cdot,\cdot\rangle_{p,S}}. \tag{55}$$

As a consequence, when $x(t)$ is not parametrized by arc length, we have

$$\kappa_n = \frac{\langle\langle x', x'\rangle\rangle_{x(t),S}}{\langle x', x'\rangle_{x(t),S}} = \frac{\mathbb{L}\,(u')^2 + 2\,\mathbb{M}\,u'\,v' + \mathbb{N}\,(v')^2}{\mathbb{E}\,(u')^2 + 2\,\mathbb{F}\,u'\,v' + \mathbb{G}\,(v')^2}. \tag{56}$$

Remark 11. From (56) we make the following crucial observation:

The normal curvature κ_n is totally determined by the surface and the tangent direction of the curve.

4.2. Properties

The second fundamental form is closely related to the first fundamental form.

LEMMA 12. *Let $w, \tilde{w} \in T_pS$. Then*

$$\langle\langle w, \tilde{w}\rangle\rangle_{p,S} = \langle \mathcal{W}_{p,S}(w), \tilde{w}\rangle_{p,S} = \langle w, \mathcal{W}_{p,S}(\tilde{w})\rangle_{p,S}. \tag{57}$$

Proof. Since $\langle\langle w, \tilde{w}\rangle\rangle_{p,S}$, $\langle \mathcal{W}_{p,S}(w), \tilde{w}\rangle_{p,S}$, and $\langle w, \mathcal{W}_{p,S}(\tilde{w})\rangle_{p,S}$ are all bilinear, it suffices to prove the following cases: $w = \sigma_u, \tilde{w} = \sigma_v$; $w = \tilde{w} = \sigma_u$; $w = \tilde{w} = \sigma_v$; $w = \sigma_v, \tilde{w} = \sigma_u$. We prove the first one and leave the other three as exercises.

We calculate

$$\langle\langle \sigma_u, \sigma_v\rangle\rangle_{p,S} = \mathbb{M}. \tag{58}$$

On the other hand, $\mathcal{W}_{p,S}(\sigma_u) = -N_u = a_{11}\,\sigma_u + a_{12}\,\sigma_v$ where

$$\begin{pmatrix} a_{11} \\ a_{12} \end{pmatrix} = \begin{pmatrix} \mathbb{E} & \mathbb{F} \\ \mathbb{F} & \mathbb{G} \end{pmatrix}^{-1} \begin{pmatrix} \mathbb{L} \\ \mathbb{M} \end{pmatrix}. \tag{59}$$

Consequently

$$\begin{aligned} \langle \mathcal{W}_{p,S}(\sigma_u), \sigma_v\rangle_{p,S} &= a_{11}\,\langle \sigma_u, \sigma_v\rangle_{p,S} + a_{12}\,\langle \sigma_v, \sigma_v\rangle_{p,S} \\ &= a_{11}\,\mathbb{F} + a_{12}\,\mathbb{G} \\ &= \begin{pmatrix} \mathbb{F} & \mathbb{G} \end{pmatrix} \begin{pmatrix} \mathbb{E} & \mathbb{F} \\ \mathbb{F} & \mathbb{G} \end{pmatrix}^{-1} \begin{pmatrix} \mathbb{L} \\ \mathbb{M} \end{pmatrix} \\ &= \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbb{L} \\ \mathbb{M} \end{pmatrix} = \mathbb{M}. \end{aligned} \tag{60}$$

Note that we have used

$$\begin{pmatrix} \mathbb{E} & \mathbb{F} \\ \mathbb{F} & \mathbb{G} \end{pmatrix} \begin{pmatrix} \mathbb{E} & \mathbb{F} \\ \mathbb{F} & \mathbb{G} \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \Longrightarrow \begin{pmatrix} \mathbb{F} & \mathbb{G} \end{pmatrix} \begin{pmatrix} \mathbb{E} & \mathbb{F} \\ \mathbb{F} & \mathbb{G} \end{pmatrix}^{-1} = \begin{pmatrix} 0 & 1 \end{pmatrix}. \tag{61}$$

The proof that $\langle \sigma_u, \mathcal{W}_{p,S}(\sigma_v)\rangle_{p,S} = \mathbb{M}$ is similar. □

5. Examples

5.1. Calculation of the second fundamental form

Example 13. Consider the unit sphere $\left(u, v, \sqrt{1-u^2-v^2}\right)$. We calculate

$$\sigma_u = \left(1, 0, \frac{-u}{\sqrt{1-u^2-v^2}}\right), \qquad \sigma_v = \left(0, 1, \frac{-v}{\sqrt{1-u^2-v^2}}\right) \tag{62}$$

which gives

$$N(u,v) = \frac{\sigma_u \times \sigma_v}{\|\sigma_u \times \sigma_v\|} = \left(u, v, \sqrt{1-u^2-v^2}\right) = \sigma(u,v). \tag{63}$$

Therefore

$$\mathbb{L}(u,v) = -\sigma_u \cdot N_u = \frac{v^2-1}{1-u^2-v^2}, \tag{64}$$

$$\mathbb{M}(u,v) = -\sigma_u \cdot N_v = \frac{-u\,v}{1-u^2-v^2}, \tag{65}$$

$$\mathbb{N}(u,v) = -\sigma_v \cdot N_v = \frac{u^2-1}{1-u^2-v^2}. \tag{66}$$

Example 14. Consider the unit sphere in spherical coordinates $(\cos u \cos v, \cos u \sin v, \sin u)$. We calculate

$$\sigma_u = (-\sin u \cos v, -\sin u \sin v, \cos u), \qquad \sigma_v = (-\cos u \sin v, \cos u \cos v, 0) \tag{67}$$

which gives

$$N(u,v) = (\cos u \cos v, \cos u \sin v, \sin u). \tag{68}$$

Therefore

$$\mathbb{L}(u,v) = -1, \tag{69}$$

$$\mathbb{M}(u,v) = 0, \tag{70}$$

$$\mathbb{N}(u,v) = -\cos^2 u. \tag{71}$$

Example 15. Consider the surface patch $\sigma(u,v) = (u, v, u^2+v^2)$. We have

$$\sigma_u = (1, 0, 2\,u), \qquad \sigma_v = (0, 1, 2\,v), \tag{72}$$

$$\sigma_{uu} = \sigma_{vv} = (0, 0, 2), \qquad \sigma_{uv} = (0, 0, 0), \tag{73}$$

and

$$N = \frac{\sigma_u \times \sigma_v}{\|\sigma_u \times \sigma_v\|} = \frac{(-2\,u, -2\,v, 1)}{\sqrt{1+4\,u^2+4\,v^2}}. \tag{74}$$

Thus we have

$$\mathbb{L} = \sigma_{uu} \cdot N = \frac{2}{\sqrt{1+4\,u^2+4\,v^2}}, \tag{75}$$

$$\mathbb{M} = \sigma_{uv} \cdot N = 0, \tag{76}$$

$$\mathbb{N} = \sigma_{vv} \cdot N = \frac{2}{\sqrt{1+4\,u^2+4\,v^2}}. \tag{77}$$

So the second fundamental form is

$$\frac{2}{\sqrt{1+4\,u^2+4\,v^2}}\,(\mathrm{d}u^2+\mathrm{d}v^2). \tag{78}$$

Exercise 9. Does this mean at any point $p\in S$, the normal curvature κ_n is a constant in every direction?

Example 16. Consider a ruled surface $\sigma(u,v)=\alpha(u)+v\,l(u)$ where $l(u)$ is of unit length. We calculate

$$\sigma_u=\alpha'(u)+v\,l'(u), \qquad \sigma_v=l(u). \tag{79}$$

This gives

$$N(u,v)=\frac{\sigma_u\times\sigma_v}{\|\sigma_u\times\sigma_v\|}=\frac{\alpha'(u)\times l(u)+v\,l'(u)\times l(u)}{\|\alpha'(u)\times l(u)+v\,l'(u)\times l(u)\|}. \tag{80}$$

We further calculate

$$\sigma_{uu}=\alpha''(u)+v\,l''(u), \quad \sigma_{uv}=l'(u), \quad \sigma_{vv}=0. \tag{81}$$

Therefore if we set $A=\|\sigma_u\times\sigma_v\|$.

$$\mathbb{L}(u,v) = \sigma_{uu}\cdot N=A^{-1}\,(\alpha''+v\,l'')\cdot(\alpha'(u)\times l(u)+v\,l'(u)\times l(u)), \tag{82}$$

$$\mathbb{M}(u,v) = \sigma_{uv}\cdot N=A^{-1}\,l'\cdot(\alpha'\times l), \tag{83}$$

$$\mathbb{N}(u,v) = \sigma_{vv}\cdot N=0. \tag{84}$$

Recalling lecture 9, we see that a ruled surface is developable if and only if $\mathbb{M}=0$.

5.2. Applications of the second fundamental form

PROPOSITION 17. *Let S be a surface whose second fundamental form is identically zero. Then S is part of a plane.*

Proof. Let σ be a surface patch for S. Then by assumption we have $N_u\cdot\sigma_u=N_u\cdot\sigma_v=0$. As N is the unit normal, naturally $N_u\cdot N=0$. Consequently $N_u=0$ as $\{\sigma_u,\sigma_v,N\}$ form a basis of $\mathbb{R}^3$. Similarly $N_v=0$. Thus N is a constant vector and therefore σ is part of a plane. $\square$

PROPOSITION 18. *Let S be a suface whose second fundamental form at every $p\in S$ is a non-zero scalar multiple of its first fundamental form at p. Then S is part of a sphere.*

Exercise 10. Prove that if S is part of a sphere, then its second fundamental form is a non-zero scalar multiple of its first fundamental form.

Proof. Let $\sigma(u,v)$ be a surface patch for S. Then there holds

$$\mathbb{L}(u,v)=c(u,v)\,\mathbb{E}(u,v), \qquad \mathbb{M}(u,v)=c(u,v)\,\mathbb{F}(u,v), \qquad \mathbb{N}(u,v)=c(u,v)\,\mathbb{G}(u,v) \tag{85}$$

for every (u,v). This leads to

$$\begin{pmatrix}\mathbb{E} & \mathbb{F}\\ \mathbb{F} & \mathbb{G}\end{pmatrix}^{-1}\begin{pmatrix}\mathbb{L} & \mathbb{M}\\ \mathbb{M} & \mathbb{N}\end{pmatrix}=c(u,v)\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}. \tag{86}$$

As a consequence, we have

$$N_u + c(u,v)\,\sigma_u = 0, \qquad N_v + c(u,v)\,\sigma_v = 0 \tag{87}$$

at every (u,v). Taking v,u derivatives of the two equations respectively, we have

$$N_{uv} + c_v\,\sigma_u + c\,\sigma_{uv} = 0 = N_{vu} + c_u\,\sigma_v + c\,\sigma_{vu} \Longrightarrow c_v\,\sigma_u = c_u\,\sigma_v. \tag{88}$$

As σ_u, σ_v form a basis of T_pS, there must hold $c_v = c_u = 0$, that is $c(u,v) = c$ is a constant.

Now (87) becomes

$$(N + c\,\sigma)_u = (N + c\,\sigma)_v = 0 \Longrightarrow N + c\,\sigma = r_0 \tag{89}$$

is a constant. In other words, we have

$$\sigma + c^{-1}\,N = c^{-1}\,r_0 \tag{90}$$

is a constant which means σ is part of the sphere centered at $c^{-1}\,r_0$ and with radius $|c|^{-1}$. $\square$