Lecture 8: The First Fundamental Form

Disclaimer. As we have a textbook, this lecture note is for guidance and supplement only. It should not be relied on when preparing for exams.

In this lecture we study how to measure distance on a surface patch.
The required textbook sections are §6.1. The optional sections are §6.2–5.

I try my best to make the examples in this note different from examples in the textbook. Please read the textbook carefully and try your hands on the exercises. During this please don't hesitate to contact me if you have any questions.

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- Review of midterm.
- Fun application of curve theory.

  Which way does the bicycle go. Very nicely explained in this youtube video: https://youtu.be/ETnbfZUW8zY. Also see http://www.mathteacherscircle.org/wp-content/themes/mtc/assets/MATH-CIRCLE-SESSION_Bicycle-Tracks.pdf for related questions.

  Exercise 1. Given the track of the rear wheel, calculate the curvature of the front wheel.

  QUESTION 1. *If all we have is a picture of the bicycle tracks (so that there is deformation), can we still determine which way the bicycle goes?*

  QUESTION 2. *What if the bicycle is on a curved surface?*

- Overview of "local theory" of surfaces.
  - "local": We are focusing on surface patches.
  - First fundamental form: Measuring distance, angle, area. $\|x'(t)\|$ for surfaces.
  - Second fundamental form: Measuring "curving". κ and τ for surfaces.
  - Turns out that the two "fundamental forms" totally determine the surface (up to a rigid motion).

    Exercise 2. Consider a curve in $\mathbb{R}^3$. Is it determined (up to a rigid motion) by $\|x'(t)\|$, $\kappa(t), \tau(t)$? Can we drop $\|x'(t)\|$ from the list?

1. Measurements on a surface patch

1.1. Motivation

- Our goal is to measure length, angle, and area on a surface patch without having to re-write everything in the ambient space $\mathbb{R}^3$.
- Recall that in $\mathbb{R}^3$, we do measurements through "inner product" of vectors: Let $w, \tilde{w}$ be vectors in $\mathbb{R}^3$, then we can compute their inner product as

$$w \cdot \tilde{w} = w_1 \tilde{w}_1 + w_2 \tilde{w}_2 + w_3 \tilde{w}_3. \tag{1}$$

  Using this we can compute length:

$$\|w\| = \sqrt{w \cdot w}, \tag{2}$$

  angle:

$$\cos\angle(w, \tilde{w}) = \frac{w \cdot \tilde{w}}{\|w\| \, \|\tilde{w}\|}, \tag{3}$$

  area of the parallelogram formed by $w, \tilde{w}$:

$$A = \|w \times \tilde{w}\| = \sqrt{\|w\|^2 \, \|\tilde{w}\|^2 - (w \cdot \tilde{w})^2}. \tag{4}$$

  Exercise 3. Prove that $\|w \times \tilde{w}\| = \sqrt{\|w\|^2 \, \|\tilde{w}\|^2 - (w \cdot \tilde{w})^2}$.

Note. Keep in mind that when we are talking about vectors in $\mathbb{R}^3$, we are talking about the $\mathbb{R}^3$ that is the "velocity overlay" space of the location space $\mathbb{R}^3$. In other words, the $\mathbb{R}^3$ here is in fact $T_p\mathbb{R}^3$. When we need to measure in the location space, we integrate:

$$L = \int_a^b \|x'(t)\| \,\mathrm{d}t, \qquad A = \int_U \|\sigma_u \times \sigma_v\| \,\mathrm{d}u \,\mathrm{d}v. \tag{5}$$

Now we ask the following natural question:

How to measure on surfaces?

Consider a point $p \in S$, a surface patch given by $\sigma: U \mapsto \mathbb{R}^3$. How should measurements be done within the tangent plane T_pS?

- Let $w, \tilde{w} \in T_pS$. Let $p = \sigma(u_0, v_0)$. Then we have

$$w = w_1\,\sigma_u(u_0, v_0) + w_2\,\sigma_v(u_0, v_0), \qquad \tilde{w} = \tilde{w}_1\,\sigma_u(u_0, v_0) + \tilde{w}_2\,\sigma_v(u_0, v_0). \tag{6}$$

  It now follows

$$w \cdot \tilde{w} = \mathbb{E}(u_0, v_0)\, w_1\,\tilde{w}_1 + \mathbb{F}(u_0, v_0)\,(w_2\,\tilde{w}_1 + w_1\,\tilde{w}_2) + \mathbb{G}(u_0, v_0)\, w_2\,\tilde{w}_2 \tag{7}$$

  where

$$\mathbb{E}(u, v) := \|\sigma_u(u, v)\|^2, \tag{8}$$
$$\mathbb{F}(u, v) := \sigma_u(u, v) \cdot \sigma_v(u, v), \tag{9}$$
$$\mathbb{G}(u, v) := \|\sigma_v(u, v)\|^2. \tag{10}$$

- The point here is that, once we know $\mathbb{E}$, $\mathbb{F}$, $\mathbb{G}$, we can do all the measurements **without leaving the surface**.

1.2. The first fundamental form

Definition 3. (Definition 6.1.1 of the textbook) *Let p be a point of a surface S. The* first fundamental form *of S at p is the bilinear form*

$$\langle w, \tilde{w} \rangle_{p,S} := w \cdot \tilde{w}. \tag{11}$$

Theorem 4. *Let $\langle \cdot, \cdot \rangle_{p,S}$ be the first fundamental form of S at p. Let σ be a surface patch of S covering p. Let $w = w_1\,\sigma_u + w_2\,\sigma_v$, $\tilde{w} = \tilde{w}_1\,\sigma_u + \tilde{w}_2\,\sigma_v$ be two vectors in T_pS. Then*

$$\langle w, \tilde{w} \rangle_{p,S} = \mathbb{E}(u_0, v_0)\, w_1\,\tilde{w}_1 + \mathbb{F}(u_0, v_0)\,(w_2\,\tilde{w}_1 + w_1\,\tilde{w}_2) + \mathbb{G}(u_0, v_0)\, w_2\,\tilde{w}_2 \tag{12}$$

where $\mathbb{E}, \mathbb{F}, \mathbb{G}$ are given by (8–10), and $p = \sigma(u_0, v_0)$.

Remark 5. There is now the natural question: What happens if we take a different first fundamental form? It can be shown that $\langle \cdot, \cdot \rangle_{p,S}$ remains the same if we choose a different surface patch for S.

Exercise 4. Try to make sense of the above statement.

Theorem 6. (Measurements on S) *Let S be a surface patch with first fundamental form $\mathbb{E}, \mathbb{F}, \mathbb{G}$. Then,*

i. if $\sigma(u(t),\ v(t))$ is a curve on S, its arc length from $t=a$ to $t=b$ is given by $\int_a^b \sqrt{\langle \sigma'(t), \sigma'(t)\rangle_{\sigma(t),S}}\, \mathrm{d}t$ which equals

$$\int_a^b \sqrt{\mathbb{E}(\sigma(t))\, u'(t)^2 + \mathbb{F}(\sigma(t))\, u'(t)\, v'(t) + \mathbb{G}(\sigma(t))\, v'(t)^2}. \tag{13}$$

ii. if $\sigma(u(t), v(t))$ and $\sigma(\tilde{u}(t), \tilde{v}(t))$ are two curves on S intersecting at $\sigma(u_0, v_0)$, then the angle between the two tangent vectors $w := \frac{\mathrm{d}\sigma(u(t), v(t))}{\mathrm{d}t}$ and $\tilde{w} := \frac{\mathrm{d}\sigma(\tilde{u}(t), \tilde{v}(t))}{\mathrm{d}t}$ is given by

$$\cos\theta = \frac{\langle w, \tilde{w}\rangle_{p,S}}{\langle w, w\rangle_{p.S}^{1/2} \langle \tilde{w}, \tilde{w}\rangle_{p,S}^{1/2}}. \tag{14}$$

iii. the area of the surface patch is given by

$$A = \int_U \sqrt{\langle w, w\rangle_{p.S} \langle \tilde{w}, \tilde{w}\rangle_{p,S} - \langle w, \tilde{w}\rangle_{p,S}^2} = \int_U \sqrt{\mathbb{E}\,\mathbb{G} - \mathbb{F}^2}\, \mathrm{d}u\, \mathrm{d}v. \tag{15}$$

1.3. First fundamental form under re-parametrization

- A common situation is the following. Say we have the first fundament form of a surface patch $\sigma(u, v)$ as $\mathbb{E}(u, v), \mathbb{F}(u, v), \mathbb{G}(u, v)$. Now the surface is re-parametrized through $u = U(\tilde{u}.\tilde{v})$ and $v = V(\tilde{u}, \tilde{v})$. How to obtain the new first fundamental form?
- The simplest way is to do the following: Substitute $u = U(\tilde{u}.\tilde{v})$, $v = V(\tilde{u}, \tilde{v})$, and

  $$\mathrm{d}u = U_{\tilde{u}}\, \mathrm{d}\tilde{u} + U_{\tilde{v}}\, \mathrm{d}\tilde{v}, \qquad \mathrm{d}v = V_{\tilde{u}}\, \mathrm{d}\tilde{u} + V_{\tilde{v}}\, \mathrm{d}\tilde{v} \tag{16}$$

  into the formula

  $$\mathbb{E}\, \mathrm{d}u^2 + 2\, \mathbb{F}\, \mathrm{d}u\, \mathrm{d}v + \mathbb{G}\, \mathrm{d}v^2 \tag{17}$$

  to obtain

  $$\tilde{\mathbb{E}}\, \mathrm{d}\tilde{u}^2 + 2\, \tilde{\mathbb{F}}\, \mathrm{d}\tilde{u}\, \mathrm{d}\tilde{v} + \tilde{\mathbb{G}}\, \mathrm{d}\tilde{v}^2. \tag{18}$$

2. Examples

Example 7. Consider the plane

$$\sigma(u, v) = (u, v, 1 + 2\, u + 3\, v). \tag{19}$$

We easily calculate

$$\sigma_u = (1, 0, 2), \qquad \sigma_v = (0, 1, 3) \tag{20}$$

which gives

$$\mathbb{E} = 5, \qquad \mathbb{F} = 6, \qquad \mathbb{G} = 10. \tag{21}$$

Thus the first fundament form is

$$5\, \mathrm{d}u^2 + 12\, \mathrm{d}u\, \mathrm{d}v + 10\, \mathrm{d}v^2. \tag{22}$$

Example 8. Consider the cylinder

$$\sigma(u,v) = (\cos u, \sin u, v). \tag{23}$$

We have

$$\sigma_u = (-\sin u, \cos u, 0), \qquad \sigma_v = (0,0,1). \tag{24}$$

This gives

$$\mathbb{E} = 1, \qquad \mathbb{F} = 0, \qquad \mathbb{G} = 1. \tag{25}$$

Thus the first fundamental form is

$$\mathrm{d}u^2 + \mathrm{d}v^2. \tag{26}$$

Example 9. Consider the unit sphere

$$\sigma(u,v) = \left(u, v, \sqrt{1-u^2-v^2}\right). \tag{27}$$

We have

$$\sigma_u = \left(1, 0, \frac{-u}{\sqrt{1-u^2-v^2}}\right), \tag{28}$$

$$\sigma_v = \left(0, 1, \frac{-v}{\sqrt{1-u^2-v^2}}\right). \tag{29}$$

This gives

$$\mathbb{E} = \frac{1-v^2}{1-u^2-v^2}, \qquad \mathbb{F} = \frac{u\,v}{1-u^2-v^2}, \qquad \mathbb{G} = \frac{1-u^2}{1-u^2-v^2} \tag{30}$$

which gives the first fundamental form as

$$\frac{1-v^2}{1-u^2-v^2}\,\mathrm{d}u^2 + 2\,\frac{u\,v}{1-u^2-v^2}\,\mathrm{d}u\,\mathrm{d}v + \frac{1-u^2}{1-u^2-v^2}\,\mathrm{d}v^2. \tag{31}$$

Example 10. Consider the torus

$$\sigma(u,v) = ((a+b\cos u)\cos v, (a+b\cos u)\sin v, b\sin u). \tag{32}$$

We have

$$\sigma_u = ((a-b\sin u)\cos v, (a-b\sin u)\sin v, b\cos u) \tag{33}$$

$$\sigma_v = (-(a+b\cos u)\sin v, (a+b\cos u)\cos v, 0). \tag{34}$$

Therefore

$$\mathbb{E} = a^2+b^2-2\,a\,b\cos u, \quad \mathbb{F} = 0, \quad \mathbb{G} = a^2+b^2\cos^2 u + 2\,a\,b\cos u. \tag{35}$$

The first fundamental form is then given by

$$(a^2+b^2-2\,a\,b\cos u)\,\mathrm{d}u^2 + (a^2+b^2\cos^2 u + 2\,a\,b\cos u)\,\mathrm{d}v^2 \tag{36}$$

We notice that $\mathbb{E}, \mathbb{F}, \mathbb{G}$ are all independent of v.

Example 11. Consider the surface patch

$$\sigma(u,v) = \left(\frac{2\,u}{u^2+v^2+1}, \frac{2\,v}{u^2+v^2+1}, \frac{u^2+v^2-1}{u^2+v^2+1}\right). \tag{37}$$

We have

$$\sigma_u = \left(\frac{2\,(v^2 - u^2 + 1)}{(u^2 + v^2 + 1)^2}, \frac{-4\,u\,v}{(u^2 + v^2 + 1)^2}, \frac{4\,u}{(u^2 + v^2 + 1)^2} \right), \tag{38}$$

$$\sigma_v = \left(\frac{-4\,u\,v}{(u^2 + v^2 + 1)^2}, \frac{2\,(u^2 - v^2 + 1)}{(u^2 + v^2 + 1)^2}, \frac{4\,v}{(u^2 + v^2 + 1)^2} \right). \tag{39}$$

Consequently we have

$$\mathbb{E} = \frac{4}{(u^2 + v^2 + 1)^2}, \tag{40}$$

$$\mathbb{F} = 0, \tag{41}$$

$$\mathbb{G} = \frac{4}{(u^2 + v^2 + 1)^2}. \tag{42}$$

Thus we see that the first fundamental form is

$$\frac{4}{(u^2 + v^2 + 1)^2}[\mathrm{d}u^2 + \mathrm{d}v^2]. \tag{43}$$

In particular, we notice that the curves $\sigma(u, v_0)$ and $\sigma(u_0, v)$ are always orthogonal at their intersection.