Lecture 5: Surfaces II

Disclaimer. As we have a textbook, this lecture note is for guidance and supplement only. It should not be relied on when preparing for exams.

In this lecture we give mathematical definition of surfaces as a compatible collection of surface patches. We also define the tangent plane and normal vectors of surfaces.

The required textbook sections are §4.2, §4.3, §4.4.

The examples in this note are mostly different from examples in the textbook. Please read the textbook carefully and try your hands on the exercises. During this please don't hesitate to contact me if you have any questions.

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1. Tangent Planes and Normal Vectors

At every point on a smooth surface, there is a unique plane "touching" the surface, called the "tangent plane" at the point. The vector at the point that is normal to the tangent point is called the "normal vector" there. We will derive the formulas for the tangent plane as well as the normal vector.

1.1. Tangent planes

- TANGENT VECTOR AND TANGENT PLANE.

DEFINITION 1. (DEFINITION 4.2.1 OF TEXTBOOK) *A surface patch $\sigma: U \mapsto \mathbb{R}^3$ is called regular if it is smooth and the vectors σ_u and σ_v are linearly independent at all points $(u, v) \in U$.*

In the following we will always assume the surface under study to have an atlas of regular surface patches. In fact, most of the times we will just focus on one single surface patch.

DEFINITION 2. (DEFINITION 4.4.1 OF TEXTBOOK) *A tangent vector to a surface S at point $p \in S$ is a tangent vector at p of a curve in S passing through p.*

When we consider all the curves in S passing through p, we obtain a collection of tangent vectors. This collection (together with the zero vector) forms a two-dimensional linear vector space called "tangent plane" of S at p. Denoted $T_p S$.

Exercise 1. Prove that if u, v are tangent vectors at p and a, b are arbitrary real numbers, then $a\,u + b\,v$ is also a tangent vector at p .

PROPOSITION 3. (PROPOSITION 4.4.2) *Let $\sigma: U \mapsto \mathbb{R}^3$ be a patch of a surface S containing a point $p \in S$, and let (u, v) be coordinates in U. The tangent space to S at p is the vector subspace of $\mathbb{R}^3$ spanned by the vectors σ_u and σ_v (the derivatives are evaluated at the point $(u_0, v_0) \in U$ such that $\sigma(u_0, v_0) = p$).*

Remark 4. In other words, we can represent the collection of tangent vectors at p as $\{a\,\sigma_u + b\,\sigma_v : a, b \in \mathbb{R}\}$.

Let $U \subseteq \mathbb{R}^2$ and $\sigma: U \mapsto \mathbb{R}^3$ be a surface patch of a surface S. Let $p = \sigma(u_0, v_0)$ for some $(u_0, v_0) \in U$. Then the tangent plane $T_p S = \{a\,\sigma_u(u_0, v_0) + b\,\sigma_v(u_0, v_0) : a, b \in \mathbb{R}\}$.

- EXAMPLES.

Example 5. (GRAPH) Let $U \subseteq \mathbb{R}^2$ and $f: U \mapsto \mathbb{R}$ be a smooth function. Then its graph $\{x_3 = f(x_1, x_2)\}$ is a surface. It is given by one surface patch $(u, v, f(u, v))$. As a consequence, we have

$$\sigma_u = (1, 0, f_u), \qquad \sigma_v = (0, 1, f_v), \tag{1}$$

and the tangent plane $T_p S$ at $p = (u_0, v_0)$ is given by

$$\{a\,(1, 0, f_u) + b\,(0, 1, f_v) : a, b \in \mathbb{R}\}. \tag{2}$$

1.2. Normal vectors and orientation

- NORMAL VECTOR.

DEFINITION 6. (NORMAL VECTOR) *A normal vector at $p \in S$ is a vector that is perpendicular to all tangent vectors at p. A unit normal vector at $p \in S$ is a normal vector at p with unit norm.*

Let $U \subseteq \mathbb{R}^2$ and $\sigma: U \mapsto \mathbb{R}^3$ be a surface patch of a surface S. Let $p = \sigma(u_0, v_0)$ for some $(u_0, v_0) \in U$. Then the normal vectors at p are given by $c\,\sigma_u \times \sigma_v$ where $c \in \mathbb{R}$. In particular, the unit normal vectors are given by

$$\pm \frac{\sigma_u \times \sigma_v}{\|\sigma_u \times \sigma_v\|}. \tag{3}$$

- ORIENTATION.
  - INFORMAL DEFINITION. A surface S is orientable if and only if there is a continuous function $N: S \mapsto \mathbb{R}^3$ such that at every $p \in S$, $N(p)$ is a unit normal vector of S at p .
  - There are surfaces that are not orientable.
  - Every regular surface patch is orientable.

2. Surface Area

The definition of surface area is subtle. However for the regular surfaces considered in 348, there is a simple formula.

2.1. How to calculate surface area

Intuitions about the surface area formula.

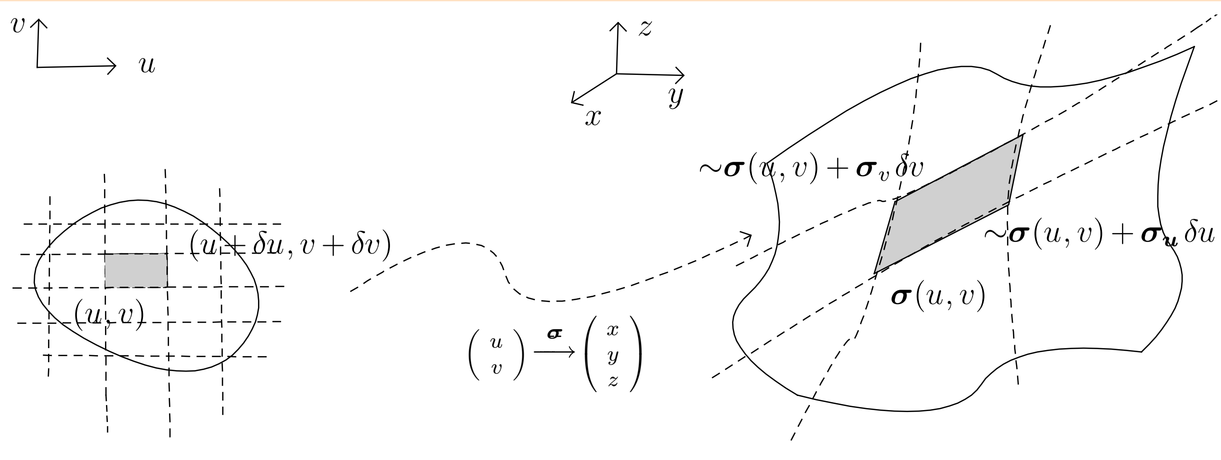

Figure 1. Stretching and twisting of of infinitesimal rectangles.

The shaded rectangle in the (u, v)-plane, with area $\delta u \cdot \delta v$, is "stretched" by the mapping $\boldsymbol{r}$ to the shaded curvilinear parallelogram in the $(x,\, y,\, z)$-space. The sides of this parallelogram are approximately $\boldsymbol{r}_u\,\delta u$ and $\boldsymbol{r}_v\,\delta v$, giving its area to be about $\|\sigma_u \times \sigma_v\|\,\delta u \cdot \delta v$. Summing the areas of all such curvilinear parallelograms up we reach the integral formula

$$\int_U \|\sigma_u \times \sigma_v\|\,\mathrm{d}u\mathrm{d}v. \tag{4}$$

Area of a surface patch:

$$\int_U \|\sigma_u \times \sigma_v\| \,\mathrm{d}u\mathrm{d}v \tag{5}$$

In particular, when the surface patch is given by a graph $z = \phi(x, y)$ on $U \subset \mathbb{R}^2$. Then

$$S = \int_U \sqrt{1 + \phi_x^2 + \phi_y^2} \,\mathrm{d}x \,\mathrm{d}y. \tag{6}$$

Example 7. Find the area of the part of $z = x\, y$ that is inside $x^2 + y^2 = 1$.

Solution. We calculate

$$S = \int_{x^2+y^2 \leqslant 1} \sqrt{1 + z_x^2 + z_y^2} \,\mathrm{d}(x, y) = \frac{2\,\pi}{3} \left(2\,\sqrt{2} - 1\right). \tag{7}$$

Example 8. Find the surface area of the sphere $x^2 + y^2 + z^2 = R^2$.

Solution. We use the parametrization

$$\sigma(\phi, \psi) = \begin{pmatrix} R\cos\phi\cos\psi \\ R\sin\phi\cos\psi \\ R\sin\psi \end{pmatrix}, \qquad U = \left\{(\phi, \psi) | \, 0 < \phi < 2\,\pi, -\frac{\pi}{2} < \psi < \frac{\pi}{2}\right\}. \tag{8}$$

Exercise 2. Note that as shown on page 72 of the textbook, one "slit" on the sphere is not covered. Convince yourself that this is not a problem for the purpose of calculating surface area. Prove that this is not a problem if you have learned the theory of Riemann integration on surfaces.

Then calculate

$$\sigma_u = \begin{pmatrix} -R\sin\phi\cos\psi \\ R\cos\phi\cos\psi \\ 0 \end{pmatrix}, \qquad \sigma_v = \begin{pmatrix} -R\cos\phi\sin\psi \\ -R\sin\phi\sin\psi \\ R\cos\psi \end{pmatrix}. \tag{9}$$

This gives

$$S = \int_D R^2 \cos\psi \,\mathrm{d}(\phi, \psi) = 4\,\pi\,R^2. \tag{10}$$

2.2. The counterexample of Schwartz (optional)

> "The example of Schwarz, ... , was the starting point of an extensive and fascinating literature. Still, we do not possess as yet a satisfactory theory of the area of surfaces, ..."
>
> —— Tibor Rado, 1943[1]

- Gelbaum, B. R. and Olmsted, J. M. H., *Counterexamples in Analysis*, Chapter 11, Example 7.

Let

$$S = \{(x, y, z) | \, x^2 + y^2 = 1, \quad 0 \leqslant z \leqslant 1\}. \tag{11}$$

Let $m \in \mathbb{N}$. Define $2\,m + 1$ circles:

$$C_{k,m} := S \cap \left\{(x, y, z) | \, z = \frac{k}{2\,m}\right\}, \qquad k = 0, 1, 2, \ldots, 2\,m. \tag{12}$$

1. Tibor Rado, *What is the Area of a Surface?*, The American Mathematical Monthly, Vol. 50, No. 3, Mar., 1943, pp. 139 - 141.

Now let $n \in \mathbb{N}$. Pick on each $C_{k,m}$ n points:

$$P_{k,m,j} := \begin{cases} \left(\cos\frac{2\,j\,\pi}{n}, \sin\frac{2\,j\,\pi}{n}, \frac{k}{2\,m}\right) & k \text{ even} \\ \left(\cos\frac{(2\,j+1)\,\pi}{n}, \sin\frac{(2\,j+1)\,\pi}{n}, \frac{k}{2\,m}\right) & k \text{ odd} \end{cases}, \qquad j = 0, 1, \ldots, n-1. \tag{13}$$

Connecting this points in a natural manner we obtain $4\,m\,n$ congruent space triangles. It can be calculated that the area of each triangle is

$$\sin\left(\frac{\pi}{n}\right)\left[\frac{1}{4\,m^2} + \left(1 - \cos\left(\frac{\pi}{n}\right)\right)^2\right]^{1/2}. \tag{14}$$

Exercise 3. Prove the above formula.

Thus the area of the polyhedron inscribed in the cylinder is

$$A_{mn} := 2\,\pi\frac{\sin(\pi/n)}{\pi/n}\left(1 + 4\,m^2\left(1 - \cos\frac{\pi}{n}\right)^2\right)^{1/2}. \tag{15}$$

Exercise 4. Prove that, as $m, n \to \infty$,

a) the diameters of the triangles $\longrightarrow 0$;

b) The limit of A_{mn} depends on how $m, n \longrightarrow \infty$. Furthermore for any $s > 2\,\pi$ (including ∞), there is a strictly increasing function $M: \mathbb{N} \mapsto \mathbb{N}$ such that

$$\lim_{n \to \infty} A_{M(n),n} = s. \tag{16}$$

Note that the area of the cylinder is $2\,\pi$.

Remark 9. See http://www.cut-the-knot.org/Outline/Calculus/SchwarzLantern.shtml for a visualization of the construction.

Remark 10. (From (Lord)) In 1868 J. A. Serret[2] suggested the "obvious" generalization of the natural method of finding arc length to calculation of surface area:

> "Given a portion of a curved surface bounded by a curve C, we call the area of this surface the limit S towards which the area of an inscribed polyhedral surface tends, where the inscribed polyhedral surface is formed by triangular faces and is bounded by the polygonal curve G, which limits the curve C"
>
> "One must show that the limit S exists and that it is independent of the way in which the faces of the inscribed surface decreases."

The problem with this approach was first realized by H. A. Schwarz[3], who wrote to Italian mathematician Gennochi about this in 1880. Later in 1882 Gennochi's student Peano annouced the same result in a course he taught. Around the same time Schwarz wrote to Hermite about his example. Hermite published Schwarz's letter in his course notes, which was published later than that of Peano's. Consequently there are disputes about priority.

2. of Frenet-Serret frame in Differential Geometry.

3. Gesammelte Mathematische Abhandlungen, Vol. 2, p. 309. Berlin, Julius Springer, 1890.

3. Differentiation of functions between surfaces

The transition from multivariable calculus to classical differential geometry is fulfilled when we start to differentiate functions mapping one surface to another. Such differentiation is defined through the help of surface patches. The differentials are linear maps between tangent planes.

- Consider two surfaces S, S'. We can consider a function f from S to S', that is given $x \in S$, we have $f(x) \in S'$ defined.[4]
- In differential geometry, we study derivatives of such maps.
- How to differentiate a function that is defined on a curved surface instead of the flat space? Flatten.
- Let $f: S \mapsto \tilde{S}$ be a function from one surface S to another surface $\tilde{S}$. Let $p \in S$. We would like to "differentiate" f at p. Remember that differentiation is local, that is we only need those values of f at x around p. Thus it suffices to "flatten" S around p. More specifically, we "pull back" S into $\mathbb{R}^2$ through a surface patch.
- The procedure.

  i. Let $\sigma: U \mapsto \mathbb{R}^3$ be a surface patch of S covering p: $\sigma(u_0, v_0) = p$. By definition of surface patches, we see that $f \circ \sigma: U \mapsto \mathbb{R}^3$ is well-defined.

  ii. Let $\tilde{\sigma}: V \mapsto \mathbb{R}^3$ be a surface patch of S' covering $f(p)$. Then we see that $F := (\tilde{\sigma})^{-1} \circ f \circ \sigma: U \mapsto V$ is a well-defined[5] function from U to V.

  iii. Let $x(t) := \sigma(u(t), v(t))$ be a curve on S with $x(t_0) = p$. Let $(\tilde{u}(t), \tilde{v}(t)) = F(u(t), v(t))$. Then the chain rule gives

  $$\begin{pmatrix} \tilde{u}' \\ \tilde{v}' \end{pmatrix} = DF(u, v) \cdot \begin{pmatrix} u' \\ v' \end{pmatrix} \tag{17}$$

  where keep in mind that DF is a matrix.

  iv. On the other hand, if we set $\tilde{x}(t) := \tilde{\sigma}(\tilde{u}(t), \tilde{v}(t))$ to be the curve in $\tilde{S}$ $x(t)$ is mapped to by f, we have

  $$x'(t_0) = u'(t_0)\, \sigma_u(p) + v'(t_0)\, \sigma_v(p) \tag{18}$$

  and

  $$\tilde{x}'(t_0) = \tilde{u}'(t_0)\, \tilde{\sigma}_u(f(p)) + \tilde{v}'(t_0)\, \tilde{\sigma}_v(f(p)). \tag{19}$$

  Thus (u', v') and $(\tilde{u}', \tilde{v}')$ are coordinates of the tangent vectors $x'(t_0)$ and $\tilde{x}'(t_0)$, and they are related by (17).

Remark 11. In fact $f \circ \sigma$ is also a surface patch on $\tilde{S}$. However in practice usually an atlas of surface patches has already been given for $\tilde{S}$ and of course those patches are usually different from $f \circ \sigma$.

4. For example, the correspondence between a map and the real locations is such a function.

5. There is a minor technical issue here. But we ignore it for now.

The differential of $f: S \mapsto \tilde{S}$ at $p \in S$, denoted $D_p f$, is a linear map between the tangent planes $T_p S$ and $T_{f(p)} \tilde{S}$. If σ and $\tilde{\sigma}$ are two surface patches on $S, \tilde{S}$ respectively, containing p, $f(p)$ respectively, then the matrix representation of $D_p f$ is the 2×2 Jacobian matrix $DF(u_0, v_0)$ where $F := (\tilde{\sigma})^{-1} \circ f \circ \sigma$, and $\sigma(u_0, v_0) = p$. In other words, we have

$$D_p f(a\,\sigma_u + b\,\sigma_v) = \tilde{a}\,\tilde{\sigma}_u + \tilde{b}\,\tilde{\sigma}_v \tag{20}$$

where all the σ_u, σ_v are evaluated at p and $\tilde{\sigma}_u, \tilde{\sigma}_v$ at $f(p)$, and

$$\begin{pmatrix} \tilde{a} \\ \tilde{b} \end{pmatrix} = DF(u_0, v_0) \cdot \begin{pmatrix} a \\ b \end{pmatrix}. \tag{21}$$

Example 12. (Stereographic projection) Let $\tilde{S}$ be the sphere $x_1^2 + x_2^2 + (x_3 - 1)^2 = 1$ taking away the north pole $(0,0,2)$. Let S be the plane $x_3 = 0$. $f: S \mapsto \tilde{S}$ be such that $(0,0,2)$, $(u,v,0)$, $f(u,v,0)$ lie on the same straight line. Then we have

$$f(u,v,0) = \left(\frac{4\,u}{u^2+v^2+4}, \frac{4\,v}{u^2+v^2+4}, \frac{2\,(u^2+v^2)}{u^2+v^2+4} \right). \tag{22}$$

Let $p = (0,0,0)$. We will calculate $D_p f$.

i. Pick σ covering p. We take $\sigma: U = \mathbb{R}^2 \mapsto \mathbb{R}^3$ defined as $\sigma(u,v) = (u,v,0)$;

ii. Pick $\tilde{\sigma}$ covering $f(p)$ and calculate $(\tilde{\sigma})^{-1}$. We calculate $f(p) = (0,0,0)$. Thus we can take

$$\tilde{\sigma}: \tilde{U} = \{\tilde{u}^2 + \tilde{v}^2 < 1\} \mapsto \mathbb{R}^3, \qquad \tilde{\sigma}(\tilde{u}, \tilde{v}) = \left(\tilde{u}, \tilde{v}, 1 - \sqrt{1 - \tilde{u}^2 - \tilde{v}^2}\right). \tag{23}$$

Thus $(\tilde{\sigma})^{-1}(x,y,z) = (x,y)$.

iii. Formulate $F = (\tilde{\sigma})^{-1} \circ f \circ \sigma$. We have

$$F = (\tilde{\sigma})^{-1} \circ f: F(u,v) = \left(\frac{4\,u}{u^2+v^2+4}, \frac{4\,v}{u^2+v^2+4} \right). \tag{24}$$

iv. Calculate $DF(u_0, v_0)$ for $\sigma(u_0, v_0) = p$. We have $\sigma(0,0) = (0,0,0) = p$ and therefore calculate

$$DF(0,0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{25}$$

The conclusion from the above calculation is that, if $v = a\,\sigma_u + b\,\sigma_v$ is a vector in $T_p S$, then $D_p f(v) \in T_{f(p)} \tilde{S}$ is given by $a\,\tilde{\sigma}_u + b\,\tilde{\sigma}_v$.

Exercise 5. Calculate DF at a different point.