Lecture 4: Surfaces I

Disclaimer. As we have a textbook, this lecture note is for guidance and supplement only. It should not be relied on when preparing for exams.

In this lecture we give mathematical definition of surfaces as a compatible collection of surface patches. We also define the tangent plane and normal vectors of surfaces.

The required textbook sections are §4.1, §4.5 (before Definition 4.5.1). The optional textbook sections are §4.5 (Definition 4.5.1 and after), §5.1, §5.2, §5.3, §5.4, §5.5, §5.6.

The examples in this note are mostly different from examples in the textbook. Please read the textbook carefully and try your hands on the exercises. During this please don't hesitate to contact me if you have any questions.

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1. Parametrization of Surfaces

Mathematical representation of an arbitrary smooth surface is non-trivial. We need to "break it up" into simple graph-like pieces, called surface patches, and then "glue" these pieces together.

1.1. The difficulties in defining surfaces mathematically

- Two naive definitions.
  - A surface is the graph of a "nice" function.

    Example 1. The graph of $f(x,y)=x^2+y^2$ defines a paraboloid.

    Example 2. It is awkward to define the unit sphere this way.

    Remark 3. This definition is too narrow.

  - A surface is the level set of a "nice" function.

    Exercise 1. Let $f(x, y)$ be a smooth function. Then there is a smooth function $F(x,y,z)$ such that the graph of $f(x,y)$ is the zero levelset $\{(x,y,z): F(x,y,z)=0\}$.

    Example 4. The unit sphere is $f(x,y,z)=0$ for $f(x,y,z)=x^2+y^2+z^2-1$.

    However, a direct consequence of Whitney's extension theorem[1] is that, any closed set in $\mathbb{R}^n$ is the zero level set of a smooth function $f(x_1,\ldots,x_n)$, that is for every closed set A, there is a smooth function f such that $A=\{f=0\}$.

1. Whitney, Hassler (1934), "Analytic extensions of functions defined in closed sets", Transactions of the American Mathematical Society, American Mathematical Society, 36 (1): 63–89, doi:10.2307/1989708.

Technical Aside

- CLOSED SET. A set $A \subseteq \mathbb{R}^n$ is closed if and only if it is the complement of an open set in $\mathbb{R}^n$.
  - Complement of a set. The complement of a set A is another set consisting of all points that are not in A. We usually denote this set by A^c.
- OPEN SET. A set $A \subseteq \mathbb{R}^n$ is open if and only if it is the union of open balls.
  - Union of sets. The union of a collection $\mathcal{W}$ of sets is another set consisting of all points that belong to at least one set in the collection. We denote the new set by $\cup_{A \in \mathcal{W}} A$.

    Exercise 2. Determine $\cup_{i=1}^{\infty} (i, i+1)$.

    Exercise 3. Determine $\cup_{k=1}^{\infty} \left(1 - \frac{1}{k}, 1 - \frac{1}{k+1}\right)$.
  - Open ball. An "open ball" in $\mathbb{R}^n$ is the set of all points $x \in \mathbb{R}^n$ satisfying $\|x - x_0\| < r$ for some $x_0 \in \mathbb{R}^n$ and $r > 0$.

  Exercise 4. The union of open sets is still an open set.

Example 5. (CANTOR SET) Consider the following set F obtained through an infinite process:

i. Take the closed interval $[0, 1]$. Drop the middle third $(1/3, 2/3)$.

ii. Take the remaining set $[0,\, 1/3] \cup [2/3,\, 1]$, drop third middle third $(1/9, 2/9), (7/9, 8/9)$.

iii. Repeat this ad infinitum.

The remaining points form a infinite closed set.

Exercise 5. Convince yourself that the Cantor set is infinite and closed.

Example 6. (SIERPINSKI CARPET) A Sierpinski carpet is an analog of Cantor set in $\mathbb{R}^2$. We start from the unit square and repreatedly take away the middle $1/9$.

Exercise 6. Describe the process more precisely.

What remains definitely does not fit our intuition of a "smooth surface", but it is the level set of a smooth function by Whitney's theorem.

Remark 7. This definition is too wide.

1.2. Surfaces

- TWO POSSIBLE FIXES OF THE SITUATION.

  i. Widen up the graph definition by allowing graphs to be "glued" together while at the same time generalize the notion of "graphs";

ii. Narrow down the level set definition.

Both are possible, but it turns out that the first approach is more convenient in most situations.

- Surface patches.
  - The generalization of "graph".
    - Motivation from curves.
      We used to think of curves as graphs of a single variable $f(x)$, but later generalize it to the "trace" of a vector function $(f_1(t), \ldots, f_n(t))$.
    - Generalization of graph.
      Instead of considering the graph of a two variable function $f(x, y)$, we consider the "trace" of the vector function $(f_1(u, v), f_2(u, v), f_3(u, v))$.
  - Surface patch.
    - A surface patch is the trace of a nice vector function $(f_1(u, v), f_2(u, v), f_3(u, v))$ over an open set $U \subseteq \mathbb{R}^2$ (meaning: a open set U in the plane).

Definition 8. (Surface patch)
A surface patch is a function $f: U \mapsto \mathbb{R}^3$ such that both f and its inverse f^{-1} are continuous, and f is bijective.

Technical Aside

- Such a function f is called a homeomorphism.
- Bijective. A function $f: X \mapsto Y$ is called bijective if
  i. f is one-to-one. That is $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$;
  ii. f is onto. That is for every $y \in Y$ there is $x \in X$ such that $f(x) = y$.
- Inverse function. When a function is bijective, we see that the mapping $y \mapsto$ the particular x such that $f(x) = y$ is also a function. We call it the inverse function of f. Denoted f^{-1}. We note that $f^{-1}: Y \mapsto X$.

Exercise 7. What goes wrong if f is not bijective?

Remark 9. A surface patch is a "mathematical deformation" of a piece of the flat plane into a curves surface in space.

Exercise 8. A graph of a function is a surface patch.

- Surfaces.

Definition 10. (Surface) *A subset S of $\mathbb{R}^3$ is a surface if, for every point $p \in S$, there is an open set U in $\mathbb{R}^2$ and a surface patch σ from $U \subseteq \mathbb{R}^2$ to S such that $p \in \sigma(U)$.*

Exercise 9. Compare with Definition 4.1.1 of Textbook.

Remark 11. Thus a surface is a subset of $\mathbb{R}^3$ that can be covered with a collection of surface patches. Such a collection is called an atlas of S.

- EXAMPLES OF SURFACES.

Example 12. (GRAPH) Let $U \subseteq \mathbb{R}^2$ and $f: U \mapsto \mathbb{R}$ be a smooth function. Then its graph $\{x_3 = f(x_1, x_2)\}$ is a surface.

Proof. Consider the surface patch $(u, v) \mapsto (u, v, f(u, v))$. □

Example 13. (SPHERE) See textbook Example 4.1.4.

Example 14. (SURFACE OF REVOLUTION) Let $f: (a, b) \mapsto \mathbb{R}$ be a smooth positive function. Its surface of revolution (around the x-axis) is defined as

$$\{(x, y, z): y^2 + z^2 = f(x)^2\}. \tag{1}$$

To see that it is a surface, consider the atlas consisting of two surface patches:

$$(u, f(u) \cos v, f(u) \sin v), \qquad (u, v) \in U := (a, b) \times (0, 2\pi) \tag{2}$$

and

$$(u, f(u) \cos v, f(u) \sin v), \qquad (u, v) \in U := (a, b) \times (-\pi, \pi). \tag{3}$$

Exercise 10. What about the surface of revolution obtained from rotating the graph of f around the y-axis? Is it a surface? Why?

Example 15. (LEVEL SURFACES) Let $f(x, y, z)$ be a smooth function. We have seen that its level surface $S := \{(x, y, z): f(x, y, z) = 0\}$ may not be a smooth surface in any reasonable sense. However we have the following result.

(THEOREM 5.1.1 OF TEXTBOOK) Assume $\nabla f(x, y, z) \neq 0$ for every $(x, y, z) \in S$. Then S is a smooth surface.