LECTURE 3: CURVES

> **Disclaimer.** As we have a textbook, this lecture note is for guidance and supplement only. It should not be relied on when preparing for exams.

> In this lecture we set up the "play ground" for the mathematical study of curves in the 2-dimensional plane and the 3-dimensional space. In particular, we will discuss appropriate mathematical representation of curves, define their tangent vectors and arc lengths, and derive formulas for these quantities.
>
> The required textbook sections are §1.1, §1.2. The optional textbook sections are §1.3, §1.4, §1.5.

> The examples in this note are different from examples in the textbook. Please read the textbook carefully and try your hands on the exercises. During this please don't hesitate to contact me if you have any questions.

TABLE OF CONTENTS

1. Parametrization of Curves

The most convenient way (for application of calculus) to represent a spatial curve mathematically is to treat it as the trajectory of a moving particle. This leads to the so-called "parametrization" of the curves. There are infinitely many possible parametrizations of the same curve.

1.1. Mathematical representations of a curve

- LEVEL CURVES. Before the advent of calculus, a curve is usually defined through level sets:

  i. (in the plane) as level sets: $f(x, y) = c$;

  ii. (in the space) as intersection of surfaces (intersection of level sets):

  $$f(x, y, z) = c_1, \qquad g(x, y, z) = c_2. \tag{1}$$

  Example 1. A circle in $\mathbb{R}^2$ is represented as

  $$(x - a)^2 + (y - b)^2 = r^2. \tag{2}$$

  A straight line in $\mathbb{R}^3$ is represented as

  $$a_1 x_1 + a_2 x_2 + a_3 x_3 = a, \qquad b_1 x_1 + b_2 x_2 + b_3 x_3 = b. \tag{3}$$

  The basic idea is to replace the study of the curve by the study of one or more simple surfaces.

- PARAMETRIZED CURVES. To apply calculus, the most convenient representation of a curve is through parametrization.[1]

  DEFINITION 2. (DEFINITION 1.1.1 IN TEXTBOOK) *A parametrized curve in $\mathbb{R}^n$ is a map[2] $x \colon (\alpha, \beta) \mapsto \mathbb{R}^n$, for some α, β with $-\infty \leqslant \alpha < \beta \leqslant \infty$.*

  Note. Recall that (a, b) means the set of all numbers $> a$ and $< b$, while $[a, b]$ means the set of all numbers $\geqslant a$ and $\leqslant b$.

  > **Exercise 1.** Do you remember/Can you guess what $(a, b]$ and $[a, b)$ means?

  Example 3. A circle in $\mathbb{R}^2$ is represented as

  $$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} a + r \cos t \\ b + r \sin t \end{pmatrix} \tag{4}$$

  for $t \in (\alpha, \beta)$ with $\alpha < 0, \beta > 2\pi$.

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[1]. The reason for such transition is that, without calculus, there is no way to study a curve directly. Instead one has to look for simple functions that could "generate" the curve and turn the study of the curves to the study of those functions.

[2]. Another name for "function".

Example 4. A straightline in $\mathbb{R}^3$ is represented as

$$\begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix} = \begin{pmatrix} x_{01} \\ x_{02} \\ x_{03} \end{pmatrix} + t \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \tag{5}$$

for $t \in (-\infty, \infty)$, or simply $x(t) = x_0 + t\,v$ for $t \in (-\infty, \infty)$.

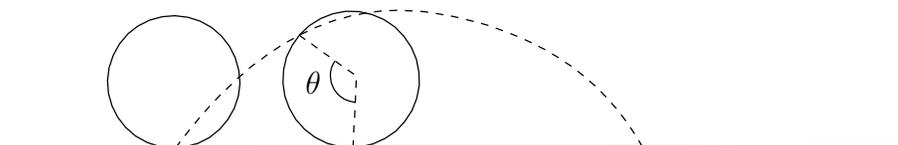**Example 5.** (CYCLOID) The trajectory of a unit circle rolling in the plane along a line.

Figure 1. The cycloids

We take $t = \theta$ as the parameter. We have $x(t) = (t - \sin t, 1 - \cos t)$ for $t \in [0, 2\pi]$.

Exercise 2. Write down a parametrized representation of a unit circle rolling outside (epicycloid) or inside (hypocycloid) another circle with radius R centered at the origin (for the "inside" case, assume $R > 1$). Plot the following two cases: Outside with $R = 1$ (Cardioid); Inside with $R = 4$ (Astroid). [3] What happens if $R \longrightarrow \infty$?

Example 6. (VIVIANI'S CURVE) The intersection of a sphere $x^2 + y^2 + z^2 = R^2$ with $x^2 + y^2 = R\,x$. It can be parametrized by

$$(R\cos^2 t, R\cos t \sin t, R\sin t), \qquad t \in [0, 2\pi) \tag{6}$$

Exercise 3. Check Ex. 1.1.8 in the textbook for what it looks like. Compare (6) with the formula there. Can you find a different parametrization?

Exercise 4. Consider a plane curve given by a graph: $y = f(x)$. Is it a level curve or a parametrized curve?

Exercise 5. Try to draw the parametrized curve $x(t) = (\cos t, \sin t, 2t)$, $t \in (\pi, 3\pi)$.

Exercise 6. Try to draw $x(t) = (t^3, t^2)$ for $t \in (-\infty, \infty)$.

Remark 7. It is clear that neither level set nor parametrized representations are unique.

- There are infinitely many pairs of planes that intersect along the same straight line;

- There are infinitely many $x(t) \colon (\alpha, \beta) \mapsto \mathbb{R}^n$ such that their image set coincide.

We emphasize that this non-uniqueness is in fact a good thing as it allows us to choose the most convenient of them. We will see how this works later.

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3. $\left(\left(\frac{1}{R} + 1 \right) \cos\left(\frac{t}{R} \right) - \frac{1}{R}\cos\left(\frac{t}{R} + t \right), \left(\frac{1}{R} + 1 \right) \sin\left(\frac{t}{R} \right) - \frac{1}{R}\sin\left(\frac{t}{R} + t \right) \right).$

In 348 we will only consider smooth curves, that is parametrized curves $x(t) = (x_1(t), x_2(t), ..., x_n(t))$, $t \in (\alpha, \beta)$, with each $x_n(t)$ infinitely differentiable for all $t \in (\alpha, \beta)$.

Exercise 7. Show through examples that the image of a smooth curve may not look smooth.

2. Tangent Vectors and Arc Length

When a curve is viewed as the trajectory of a moving particle, its velocities are represented mathematically as "tangent vectors", and the length of the trajectory is the "arc length".

2.1. From possible velocities to angent vectors

- FROM POSSIBLE VELOCITIES TO TANGENT VECTORS.

  ○ Let $x(t) \in \mathbb{R}^n$ be the trajectory of a particle. Then its velocity at t_0 is given by

  $$v(t_0) = \lim_{t \longrightarrow t_0} \frac{x(t) - x(t_0)}{t - t_0} = \frac{\mathrm{d}x}{\mathrm{d}t}(t_0). \tag{7}$$

  ○ For a parametrized curve $x(t)$, we will view it as a mathematical model of a trajectory of a particle and define its velocity at $x(t_0)$ to be one tangent vector at $x(t_0)$.

  DEFINITION 8. (TANGENT VECTOR) *Let C be a curve in $\mathbb{R}^n$. Let $x_0 \in C$. Then $v \in \mathbb{R}^n$ is a tangent vector of C at x_0 if and only if there is a parametrization $x(t): (\alpha, \beta) \mapsto \mathbb{R}^n$ of C such that*

   i. there is $t_0 \in (\alpha, \beta)$ with $x(t_0) = x_0$;

   ii. $x'(t_0) = v$.

  Exercise 8. Let C be a curve in $\mathbb{R}^n$. Let $x_0 \in C$. Let v be a tangent vector of C at x_0.

   i. Prove that $-v$ is also a tangent vector of C at x_0;

   ii. Prove that $2v$ is also a tangent vector of C at x_0;

   iii. Convince yourself that for arbitrary $c \in \mathbb{R}$, cv is a tangent vector of C at x_0;

   iv. (optional) Prove that for arbitrary $c \in \mathbb{R}$, cv is a tangent vector of C at x_0.

  A tangent vector at $x(t)$: $x'(t)$

- REGULAR CURVES.

  ○ A parametrized curve $x(t): (\alpha, \beta) \mapsto \mathbb{R}^n$ is regular if and only if it is smooth and $x'(t) \neq 0$ for all $t \in (\alpha, \beta)$.

  Example 9. The parametrized curve $x(t) = \begin{pmatrix} t \\ t^2 \\ t^3 \end{pmatrix}$ is regular. To see this we check

   i. t, t^2, t^3 are all smooth;

ii. $x'(t) = \begin{pmatrix} 1 \\ 2t \\ 3t^2 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ no matter what t is.

From now on "curve" means "regular curve".

2.2. Arc length of a parametrized curve

- THE ARC LENGTH FORMULA.

  ○ Let $x(t) : (\alpha, \beta) \mapsto \mathbb{R}^n$ be a parametrized curve. Let $[a, b] \subset (\alpha, \beta)$. We try to obtain its arc length L through the following: Pick $a = t_0 < \cdots < t_k = b$. Consider the straight line segments $\overline{x(t_0)x(t_1)}$, $\overline{x(t_1)x(t_2)}$, $\overline{x(t_2)x(t_3)}$, ... , $\overline{x(t_{k-1})x(t_k)}$. Then intuitively we have $L \geqslant$ the sum of the lengths of these line segments.

  ○ Inspired by the above, one defines

  $$L := \sup_{a=t_0<\cdots<t_k=b} \sum_{i=0}^{k-1} \|x(t_i) - x(t_{i-1})\| \tag{8}$$

  where we emphasize that the supreme is taken over all possible partitions $a = t_0 < \cdots < t_k = b$. In particular, $k \in \mathbb{N}$ is not fixed.

  ○ The formula.

  $$\text{Arc length from } x(a) \text{ to } x(b) = \int_a^b \|x'(t)\| \, \mathrm{d}t.$$

  THEOREM 10. *Let $x(t)$ be a smooth parametrized curve. Then its arc length from $x(a)$ to $x(b)$ is given by*

  $$L = \int_a^b \|x'(t)\| \, \mathrm{d}t. \tag{9}$$

  Proof. (OPTIONAL; YOU MAY WANT TO READ THE BOX AFTER THE PROOF FIRST) We prove it in two steps.

  1. $L \leqslant \int_a^b \|x'(t)\| \, \mathrm{d}t$.
     By fundamental theorem of calculus, we have

     $$x(t_i) - x(t_{i-1}) = \int_{t_{i-1}}^{t_i} x'(t) \, \mathrm{d}t. \tag{10}$$

     Thus we have

     $$\begin{aligned}
     \sum_{t=0}^{k-1} \|x(t_i) - x(t_{i-1})\| &= \sum_{t=0}^{k-1} \left\| \int_{t_{i-1}}^{t_i} x'(t) \, \mathrm{d}t \right\| \\
     &\leqslant \sum_{t=0}^{k-1} \int_{t_{i-1}}^{t_i} \|x'(t)\| \, \mathrm{d}t \\
     &= \int_a^b \|x'(t)\| \, \mathrm{d}t. \tag{11}
     \end{aligned}$$

2. $L \geqslant \int_a^b \|x'(t)\| \, dt$.

Let $k \in \mathbb{N}$ and define $t_i := a + \frac{i}{k}(b-a)$. Then we have $a = t_0 < t_1 < \cdots < t_k = b$.

By fundamental theorem of calculus, we have

$$x(t_i) - x(t_{i-1}) = \int_{t_{i-1}}^{t_i} x'(t) \, dt = x'(t_{i-1})(t_i - t_{i-1}) + R_i \tag{12}$$

where

$$R_i := \int_{t_{i-1}}^{t_i} [x'(t) - x'(t_{i-1})] \, dt = \int_{t_{i-1}}^{t_i} \left[\int_{t_{i-1}}^{t} x''(s) \, ds \right] dt. \tag{13}$$

As $x(t)$ is a regular curve, there is $M > 0$ such that

$$\|x''(s)\| \leqslant M \qquad \text{for all } s \in [a, b]. \tag{14}$$

Consequently

$$\|R_i\| \leqslant \int_{t_{i-1}}^{t_i} \int_{t_{i-1}}^{t_i} M \, ds \, dt = M(t_i - t_{i-1})^2 = \frac{M(b-a)^2}{k^2}. \tag{15}$$

Thus we have

$$\begin{aligned}
\sum_{i=0}^{k-1} \|x(t_i) - x(t_{i-1})\| &= \sum_{i=0}^{k-1} \|x'(t_{i-1})(t_i - t_{i-1}) + R_i\| \\
&\geqslant \sum_{i=0}^{k-1} [\|x'(t_{i-1})(t_i - t_{i-1})\| - \|R_i\|] \\
&\geqslant \sum_{i=0}^{k-1} \|x'(t_{i-1})\|(t_i - t_{i-1}) - \frac{M(b-a)^2}{k^2}. \tag{16}
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
\int_{t_{i-1}}^{t_i} \|x'(t)\| \, dt &= \int_{t_{i-1}}^{t_i} [\|x'(t)\| - \|x'(t_{i-1})\|] \, dt + \\
&\quad + \|x'(t_{i-1})\|(t_i - t_{i-1}) \\
&\leqslant \|x'(t_{i-1})\|(t_i - t_{i-1}) + \int_{t_{i-1}}^{t_i} \|x'(t) - x'(t_{i-1})\| \, dt \\
&= \|x'(t_{i-1})\|(t_i - t_{i-1}) + \\
&\quad + \int_{t_{i-1}}^{t_i} \left\| \int_{t_{i-1}}^{t} \|x''(s)\| ds \right\| dt \\
&\leqslant \|x'(t_{i-1})\|(t_i - t_{i-1}) + M(t_i - t_{i-1})^2 \\
&= \|x'(t_{i-1})\|(t_i - t_{i-1}) + \frac{M(b-a)^2}{k^2}. \tag{17}
\end{aligned}$$

From this we conclude

$$\sum_{i=0}^{k-1} \|x(t_i) - x(t_{i-1})\| - \int_a^b \|x'(t)\| \, \mathrm{d}t \geqslant -\frac{2\,M\,(b-a)^2}{k^2} \longrightarrow 0 \qquad (18)$$

as $k \longrightarrow \infty$. This implies

$$L := \sup_{a=t_0<t_1<\cdots<t_k=b} \sum_{i=0}^{k-1} \|x(t_i) - x(t_{i-1})\| \geqslant \int_a^b \|x'(t)\| \, \mathrm{d}t \qquad (19)$$

and ends the proof. $\qquad\qquad\square$

Technical Aside

In the above proof we have used the following results from calculus.

– SUPREME. Let A be a collection (set) of numbers. Then its supreme $\sup A$ is the smallest number that is greater than or equal to A. As a consequence,

- to prove that $\sup A \leqslant b$, all we need to do is to show that for every $a \in A$, there holds $a \leqslant b$;

- to prove that $\sup A \geqslant b$, all we need to do is to find one particular sequence of $a_n \in A$ such that $\lim_{n\to\infty} (a_n - b) \geqslant 0$.

  Exercise 9. What do we need to do to prove $\sup A > b$ or $\sup A < b$?

– TRIANGLE INEQUALITY.

- Classical triangle inequality:

$$\|x\| + \|y\| \geqslant \|x + y\|. \qquad (20)$$

- Variant:

$$\big|\|x\| - \|y\|\big| \leqslant \|x - y\|. \qquad (21)$$

- Generalization:

$$\|x_1\| + \cdots + \|x_k\| \geqslant \|x_1 + \cdots + x_k\|. \qquad (22)$$

- Integral: Together with the definite of Riemann integrals, (22) yields the following inequality for vector functions:

$$\int_a^b \|x(t)\| \, \mathrm{d}t \geqslant \left\| \int_a^b x(t) \, \mathrm{d}t \right\|. \qquad (23)$$

- Examples.

Example 11. Calculate the circumference of the unit circle $x^2 + y^2 = 1$.

Differential Geometry of Curves & Surfaces

Solution.

- Method 1. We calculate the curve length l of the graph $y = \sqrt{1-x^2}, -1 \leqslant x \leqslant 1$. Then the circumference is $L = 2\,l$.

$$
\begin{aligned}
l &= \int_{-1}^{1} \sqrt{1 + \left[\left(\sqrt{1-x^2}\right)'\right]^2}\, \mathrm{d}x \\
&= \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}}\, \mathrm{d}x \\
&\underset{\underline{x = \sin t}}{=} \int_{-\pi/2}^{\pi/2} \mathrm{d}t = \pi.
\end{aligned}
\tag{24}
$$

So the circumference is $L = 2\,\pi$.

- Method 2. We parametrize $x(t) = \cos t,\, y(t) = \sin t, 0 \leqslant t < 2\,\pi$. Then

$$
L = \int_0^{2\pi} \sqrt{x'(t)^2 + y'(t)^2}\, \mathrm{d}t = 2\,\pi.
\tag{25}
$$

Example 12. Calculate the arc length of the space curve

$$
x = \cos t,\, y = \sin t,\, z = t
\tag{26}
$$

for t from 0 to $2\,\pi$.

Solution. We have

$$
\begin{aligned}
L &= \int_0^{2\pi} \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}\, \mathrm{d}t \\
&= \int_0^{2\pi} \sqrt{2}\, \mathrm{d}t = 2\sqrt{2}\,\pi.
\end{aligned}
\tag{27}
$$

Example 13. Calculate the arc length of the cycloid $(t - \sin t, 1 - \cos t)$ from $t = 0$ to $2\,\pi$.

Solution. We have

$$
\begin{aligned}
L &= \int_0^{2\pi} \sqrt{(1 - \cos t)^2 + (\sin t)^2}\, \mathrm{d}t \\
&= \int_0^{2\pi} \sqrt{2 - 2\cos t}\, \mathrm{d}t \\
&= \int_0^{2\pi} \sqrt{2 - 2\left(1 - 2\left(\sin\frac{t}{2}\right)^2\right)}\, \mathrm{d}t \\
&= 2 \int_0^{2\pi} \left|\sin\left(\frac{t}{2}\right)\right|\, \mathrm{d}t \\
&= 8.
\end{aligned}
\tag{28}
$$

Exercise 10. Calculate the arc length of $x = \cos^3 t,\, y = \sin^3 t, t \in [0, 2\,\pi)$.

Exercise 11. (OPTIONAL) It is clear that arc length should be independent of parametrization. Prove this.

3. Arc Length Parametrization

> Among the infinitely many possible parametrizations of the same curve, one
> particular parametrization, called "arc length parametrization" stands out as
> the most convenient due to its ability to simplify calculations.

- MOTIVATION. For a general parametrized curve $x(t)\colon (\alpha,\,\beta) \mapsto \mathbb{R}^n$, in general we
  have $\|x'(t)\|$ varying with t. As we will see later, many calculations could be greatly
  simplified if $\|x'(t)\| = 1$ for all $t \in (a,\,b)$. Such a parametrization is called the "arc
  length" parametrization of the curve.

    Exercise 12. In this case we have the arc length between $x(t_0)$ and $x(t_1)$ to be $t_1 - t_0$.

- EXISTENCE OF ARC LENGTH PARAMETRIZATION.

  THEOREM 14. *Let $x(t)\colon (\alpha,\,\beta) \mapsto \mathbb{R}^n$ be a regular curve. Then there is a strictly
  increasing function $T\colon (a,b) \mapsto (\alpha,\beta)$ such that the curve $X(s) := x(T(s))$ is parametrized by its arc length.*

> NOTATION. *The convention is that, when the parametrization is arc length parametrization, we use s as the variable. That is, when we write a curve as $x(s)$, we
> assume it is already parametrized by arc length.*

  Proof. (OPTIONAL) See Proposition 1.3.6 in the textbook. $\qquad\square$

- Examples.

  Example 15. Consider the circle $(2\cos t, 2\sin t)$ in the plane. We have

  $$\|x'(t)\| = \|(-2\sin t, 2\cos t)\| = 2. \tag{29}$$

  Thus the arc length parametrization is $(2\cos(s/2), 2\sin(s/2))$.

  Example 16. Consider the space curve

  $$x = \cos t,\, y = \sin t,\, z = t. \tag{30}$$

  The arc length parametrization is $\left(\cos\left(s/\sqrt{2}\right), \sin\left(s/\sqrt{2}\right), s/\sqrt{2}\right)$.

4. Shortest Distances

4.1. On the plane

- THE PROBLEM.
  Let y, z be two points in $\mathbb{R}^n$. There are infinitely many curves that connect the
  two points. Find the one with shortest arc length.

- MATHEMATICAL FORMULATION.
  Among all curves $x(t)$ satisfying $x(a) = y,\ x(b) = z$, find the one with minimal

  $$L := \int_a^b \|x'(t)\|\, \mathrm{d}t. \tag{31}$$

Differential Geometry of Curves & Surfaces

- THE SOLUTION.
  We notice that

  i. There holds
  $$L \geqslant \left\| \int_a^b x'(t) \, \mathrm{d}t \right\| = \|x(b) - x(a)\| = \|y - z\|. \tag{32}$$

  ii. For the curve $x_L(t) = \frac{b-t}{b-a} y + \frac{t-a}{b-a} z$, we have
  $$L = \int_a^b \|x'_L(t)\| \, \mathrm{d}t = \int_a^b \left\| \frac{1}{(b-a)} (z - y) \right\| \, \mathrm{d}t = \|y - z\|. \tag{33}$$

  Therefore the solution is $x_L(t)$ which is a straight line.

  Exercise 13. Prove that $x_L(t)$ is a straight line and find its arc length parametrization.

4.2. On the cylinder

- THE PROBLEM.
  Let $y = (y_1, y_2, y_3), z = (z_1, z_2, z_3)$ be two points in $\mathbb{R}^3$ lying on the cylinder with the base circle centered at the original and with radius 1. Find the shortest path along the cyclinder surface connecting the two points.

- MATHEMATICAL FORMULATION.
  Among all curves $x(t) = (\cos t, \sin t, x_3(t))$ satisfying $x(a) = y, x(b) = z$, find the one with minimal
  $$L := \int_a^b \|x'(t)\| \, \mathrm{d}t. \tag{34}$$

- THE SOLUTION. (OPTIONAL)
  We have
  $$L = \int_a^b \sqrt{1 + x'_3(t)^2} \, \mathrm{d}t \tag{35}$$

  with $x_3(t)$ satisfying $x_3(a) = y_3, x_3(b) = z_3$ and also $\cos a = y_1, \cos b = z_1$; $\sin a = y_2$, $\sin b = z_2$.

  Since $x'_3(t)$ is a smooth function, we have
  $$L = \lim_{k \to \infty} \frac{b-a}{k} \sum_{i=0}^{k-1} \sqrt{1 + x'_3(t_i)^2} \tag{36}$$

  where $t_i = a + \frac{i}{k}(b - a)$. Now notice that the function $f(x) := \sqrt{1 + x^2}$ is convex, by Jensen's inequality we have
  $$\frac{1}{k} \sum_{i=0}^{k-1} \sqrt{1 + x'_3(t_i)^2} \geqslant \sqrt{1 + \left(\frac{1}{k} \sum_{i=0}^{k-1} x'_3(t_i) \right)^2}. \tag{37}$$

  As
  $$\lim_{k \to \infty} \frac{b-a}{k} \sum_{i=0}^{k-1} x'_3(t_i) = \int_a^b x'_3(t) \, \mathrm{d}t = z_3 - y_3 \tag{38}$$

we have

$$\begin{aligned}
L &\geqslant (b-a)\sqrt{1+\left(\frac{z_3-y_3}{b-a}\right)^2} \\
&= \sqrt{(b-a)^2+(z_3-y_3)^2} \\
&\geqslant \sqrt{(b_0-a_0)^2+(z_3-y_3)^2}.
\end{aligned} \tag{39}$$

where $b_0 - a_0 < \pi$ and satisfies also $\cos a_0 = y_1$, $\cos b_0 = z_1$; $\sin a_0 = y_2$, $\sin b_0 = z_2$.

On the other hand, the arc length of the curve

$$\left(\cos t, \sin t, y_3 + \frac{t-a_0}{b_0-a_0}(z_3-y_3)\right) \tag{40}$$

is exactly $\sqrt{(b_0-a_0)^2+(z_3-y_3)^2}$.

Exercise 14. Visualize this shortest path. What would it look like if we "flatten" the cylinder?

4.3. On the sphere

- THE PROBLEM.

    Let $y = (y_1, y_2, y_3)$, $z = (z_1, z_2, z_3)$ be two points on unit sphere centering at the origin. Find the shortest path on the sphere connecting them.

- MATHEMATICAL FORMULATION.

    Let $y = (y_1, y_2, y_3)$, $z = (z_1, z_2, z_3)$ satisfy $y_1^2 + y_2^2 + y_3^2 = z_1^2 + z_2^2 + z_3^2 = 1$. Among all curves with $x(a) = y$, $x(b) = z$, $x_1^2(t) + x_2^2(t) + x_3^2(t) = 1$, find the one minimizing the integral

$$L := \int_a^b \|x'(t)\| \, dt. \tag{41}$$

- THE SOLUTION. (OPTIONAL)

    Our goal is to show that the minimizing curve is the great arc connecting y, z. Due to the symmetry of the problem, it suffices to prove this for $y = (0, 1, 0)$ and $z = (0, 0, 1)$.

    Exercise 15. Why is this so?

    Thus all we need to show is $L \geqslant \pi/2$.

    For arbitrary $x(t)$ on the sphere connecting y, z, we define a new curve:

$$X(t) = (0, r(t), x_3(t)) \tag{42}$$

where $r(t) := (x_1(t)^2 + x_2(t)^2)^{1/2}$. We notice that $X(t)$ connects y, z and covers the great arc connecting y, z. Therefore the arc length of $X(t)$ is no less than $\pi/2$. For $X(t)$ we calculate

$$\begin{aligned}
\frac{\pi}{2} \leqslant L_X &= \int_a^b \sqrt{r'(t)^2 + x_3'(t)^2} \, dt \\
&= \int_a^b \sqrt{\frac{(x_1(t)\, x_1'(t) + x_2(t)\, x_2'(t))^2}{x_1(t)^2 + x_2(t)^2} + x_3'(t)^2} \, dt \\
&\leqslant \int_a^b \sqrt{x_1'(t)^2 + x_2'(t)^2 + x_3'(t)^2} \, dt = L.
\end{aligned} \tag{43}$$

Technical Aside

CAUCHY-SCHWARTZ. The crucial step, the one with $\leqslant$, is due to the so-called Cauchy-Schwartz inequality for vectors:

$$|u \cdot v| \leqslant \|u\| \, \|v\|. \tag{44}$$

To prove it, notice that

$$(u - t\,v) \cdot (u - t\,v) = \|u - t\,v\|^2 \geqslant 0 \tag{45}$$

for all t. Expanding the left hand side we see that

$$\|v\|^2 \, t^2 - 2\,(u \cdot v)\,t + \|u\|^2 \geqslant 0 \tag{46}$$

holds for all t, and the conclusion follows.

Exercise 16. Finish the proof. (Hint: When is a quadratic equation $t^2 - A\,t + B = 0$ having at most one real solution?)

Remark 17. From the above we see that for each case (plane, cyclinder, sphere), a new idea/technique is needed, worse still, we have to somehow know the answer before we start– much the same as the situation in classical geometry. **The reason for this is that we have used too little calculus**. In the following weeks, we will apply more calculus to geometry problems, and eventually develop a complete theory for the problem "finding shortest path on a surface". In this theory, the discovery of such path will be reduced to the solution of a single set of ODEs whose derivation is mechanical.

NO MORE AD HOC IDEAS, NO MORE CLEVER TRICKS.