Homework 9: Gauss-Bonnet

(Total 20 pts; Due Dec. 2 12pm)

Question 1. (5 pts) *Let S be a regular, orientable, compact surface with positive Gaussian curvature: $K > K_{\min} > 0$. Prove that the surface area of S is less than $4\pi / K_{\min}$.*

Proof. Take any simple closed curve γ on S. γ divides S into two regions Ω_1, Ω_2. Let γ be oriented such that Ω_1 is its interior. Then by Gauss-Bonnet theorem

$$\int_{\Omega_1} K \, \mathrm{d}S + \int_\gamma \kappa_g \, \mathrm{d}s = 2\pi, \qquad \int_{\Omega_2} K \, \mathrm{d}S + \int_{-\gamma} \kappa_g \, \mathrm{d}s = 2\pi \tag{1}$$

where $-\gamma$ is γ with the opposite orientation. Since

$$\int_{-\gamma} \kappa_g \, \mathrm{d}s = -\int_\gamma \kappa_g \, \mathrm{d}s \tag{2}$$

we have

$$4\pi = \int_S K \, \mathrm{d}S \geqslant \int_S K_{\min} \, \mathrm{d}S \tag{3}$$

and the conclusion follows. □

Question 2. (5 pts) *Let S be a compact oriented surface that can be smoothly deformed into a sphere. Let γ be a simple closed geodesic separating S into two regions A, B. Let $\mathcal{G}$: $S \mapsto \mathbb{S}^2$ be the Gauss map. Prove that $\mathcal{G}(A)$ and $\mathcal{G}(B)$ have the same area.*

Proof. Since $\mathbb{S}^2$ taking away one point can be covered by one single surface patch, so can S. Let $\sigma(u,v)$ be such a surface patch for S. Then we have

$$\int_S K \, \mathrm{d}S = \int_U K(u,v) \sqrt{\mathbb{E}\,\mathbb{G} - \mathbb{F}^2} \, \mathrm{d}u \, \mathrm{d}v. \tag{4}$$

Now let U_A, U_B be such that $\sigma(U_A) = A$, $\sigma(U_B) = B$ (maybe missing one point). Denote $N(u,v) := \mathcal{G}(\sigma(u,v))$. Then we have

$$\text{Area of } \mathcal{G}(A) = \int_{U_A} \|N_u \times N_v\| \, \mathrm{d}u \, \mathrm{d}v. \tag{5}$$

Recalling

$$-N_u = a_{11}\,\sigma_u + a_{12}\,\sigma_v, \qquad -N_v = a_{21}\,\sigma_u + a_{22}\,\sigma_v, \tag{6}$$

we have

$$N_u \times N_v = \det\begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix} \sigma_u \times \sigma_v = K\,\sigma_u \times \sigma_v. \tag{7}$$

Consequently

$$\begin{aligned} \int_{U_A} \|N_u \times N_v\| \, \mathrm{d}u \, \mathrm{d}v &= \int_{U_A} K \, \|\sigma_u \times \sigma_v\| \, \mathrm{d}u \, \mathrm{d}v \\ &= \int_{U_A} K(u,v) \sqrt{\mathbb{E}\,\mathbb{G} - \mathbb{F}^2} \, \mathrm{d}u \, \mathrm{d}v \\ &= \int_A K \, \mathrm{d}S \\ &= 2\pi - \int_\gamma \kappa_g \, \mathrm{d}s = 2\pi. \end{aligned} \tag{8}$$

Similarly we have Area of $\mathcal{G}(B) = 2\pi$. □

QUESTION 3. *Let S be a developable surface. Let γ be a curve on S. Let $\tilde{\gamma}$ be the curve corresponding to γ on the plane that is the "flattened" S. Prove or disprove: The geodesic curvature of γ and the signed curvature of $\tilde{\gamma}$ are the same at corresponding points.*

Solution. We prove that the claim is true.

Let $\sigma(u,v)\colon U \mapsto S$ a local isometry from the plane to S. Clearly $\sigma(u,v)$ can serve as a surface patch. Furthermore we have $\mathbb{E} = \mathbb{G} = 1, \mathbb{F} = 0$ and consequently all $\Gamma_{ij}^k = 0$. Note that this implies the surface normal

$$N = \sigma_u \times \sigma_v, \tag{9}$$

and that $\sigma_{uu}, \sigma_{uv}, \sigma_{vv} \parallel N$.

Now let $(u(s), v(s))$ be an arc length parametrization of $\tilde{\gamma}$. We then see that $x(s) := \sigma(u(s), v(s))$ is an arc length parametrization of γ. Thus

$$\begin{aligned}
\kappa_g &= x'' \cdot (N \times x') \\
&= [\sigma_{uu}(u')^2 + 2\sigma_{uv} u' v' + \sigma_{vv}(v')^2 + \sigma_u u'' + \sigma_v v''] \cdot [(\sigma_u \times \sigma_v) \cdot (u' \sigma_u + v' \sigma_v)] \\
&= [\sigma_{uu}(u')^2 + 2\sigma_{uv} u' v' + \sigma_{vv}(v')^2 + \sigma_u u'' + \sigma_v v''] \cdot (u' \sigma_v - v' \sigma_u) \\
&= v'' u' - u'' v' \\
&= \begin{pmatrix} u \\ v \end{pmatrix}'' \cdot \left[\begin{pmatrix} u \\ v \end{pmatrix}' \right]^{\perp} = \kappa_s.
\end{aligned} \tag{10}$$

QUESTION 4. (5 PTS) *Let $f\colon S_1 \mapsto S_2$ be a local isometry. Let a curve $\gamma_1 \subset S_1$ and $\gamma_2 := f(\gamma_1)$. Let $w_1(s)$ be a parallel tangent vector field along γ_1. For every $p \in \gamma_1$, Let $w_2(f(p)) := (Df)(p)(w_1(p))$. Then $w_2(s)$ is a tangent vector field along γ_2. Prove or disprove: w_2 is parallel along γ_2.*

Solution. We prove that the claim is true.

Let $\sigma_1(u,v)$ be a surface patch for S_1 and let $\sigma_2(u,v) := f(\sigma_1(u,v))$. Also let $x_1(s)$ be an arc length parametrization of γ_1 and let $x_2(s) := f(x_1(s))$. Since f is a local isometry, s is also the arc length parameter of γ_2.

In this setup we have $\sigma_{2,u} = (Df)(\sigma_{1,u})$ and $\sigma_{2,v} = (Df)(\sigma_1, v)$. Now let $w_1(s) = \alpha(s)\,\sigma_{1,u} + \beta(s)\,\sigma_{1,v}$. Then we have $w_2(s) = \alpha(s)\,\sigma_{2,u} + \beta(s)\,\sigma_{2,v}$. Since $w_1(s)$ is parallel along γ_1, we have

$$\begin{aligned}
(\mathbb{E}_1 \alpha + \mathbb{F}_1 \beta)' &= \frac{1}{2} \begin{pmatrix} \alpha & \beta \end{pmatrix} \begin{pmatrix} \mathbb{E}_1 & \mathbb{F}_1 \\ \mathbb{F}_1 & \mathbb{G}_1 \end{pmatrix}_u \begin{pmatrix} u' \\ v' \end{pmatrix}, \\
(\mathbb{F}_1 \alpha + \mathbb{G}_1 \beta)' &= \frac{1}{2} \begin{pmatrix} \alpha & \beta \end{pmatrix} \begin{pmatrix} \mathbb{E}_1 & \mathbb{F}_1 \\ \mathbb{F}_1 & \mathbb{G}_1 \end{pmatrix}_v \begin{pmatrix} u' \\ v' \end{pmatrix}.
\end{aligned} \tag{11}$$

But since $\mathbb{E}_1 = \mathbb{E}_2, \mathbb{F}_1 = \mathbb{F}_2, \mathbb{G}_1 = \mathbb{G}_2$, $(\alpha(s), \beta(s))$ satisfies the corresponding equations on S_2 and consequently w_2 is also parallel along γ_2.

The following are more abstract or technical questions. They carry bonus points.

There is no bonus problem for this homework.